Published by Institute of Physics Publishing for SISSA

Received: *June 5, 2008*
Accepted: *July 21, 2008*
Published: *August 5, 2008*

M5-brane in three-form flux and multiple M2-branes

Pei-Ming Ho

Department of Physics and Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan, R.O.C.
E-mail: pmho@phys.ntu.edu.tw

Yosuke Imamura, Yutaka Matsuo and Shotaro Shiba

Department of Physics, Faculty of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan
E-mail: imamura@hep-th.phys.s.u-tokyo.ac.jp, matsuo@phys.s.u-tokyo.ac.jp, shiba@hep-th.phys.s.u-tokyo.ac.jp

Abstract: We investigate the Bagger-Lambert-Gustavsson model associated with the Nambu-Poisson algebra as a theory describing a single M5-brane. We argue that the model is a gauge theory associated with the volume-preserving diffeomorphism in the three-dimensional internal space. We derive gauge transformations, actions, supersymmetry transformations, and equations of motions in terms of six-dimensional fields. The equations of motions are written in gauge-covariant form, and the equations for tensor fields have manifest self-dual structure. We demonstrate that the double dimensional reduction of the model reproduces the non-commutative $\mathrm{U}(1)$ gauge theory on a D4-brane with a small non-commutativity parameter. We establish relations between parameters in the BLG model and those in M-theory. This shows that the model describes an M5-brane in a large C-field background.

Keywords: p-branes, Non-Commutative Geometry, M-Theory.

JHEP08(2008)014

Contents

1. Introduction

Recently, a model of the M5-brane world-volume field theory was constructed [1] as a system of infinitely many M2-branes. The theory of Bagger, Lambert [2] and Gustavsson [3] was used to describe the multiple M2-brane system. In the BLG model, a background configuration of the M2-brane system corresponds to the choice of a Lie 3-algebra [4], and the Lie 3-algebra used for the M5-brane is the Nambu-Poisson algebra [5] on a 3-manifold $\mathcal{N}$ which appears as the internal space from the M2-branes' point of view, but it constitutes the M5-brane world-volume together with the M2-brane world-volume.

It was shown [1] that at the quadratic order of the Lagrangian, the M5-brane theory contains a self-dual two-form gauge field, in addition to the scalars corresponding to fluctuations of the M5-brane in the transverse directions, as well as their fermionic super-partners. To the order that was computed, this M5-brane is different from, but compatible with previous formulations of the M5-brane theory [6, 7]. In [1] , higher order terms of the Lagrangian were not considered, and a truncation was applied as a short-cut to show the desired properties of the M5-brane model. In this paper, we show that actually the "truncation" did not really remove any physical degrees of freedom. The only physical components of the gauge field are exactly those surviving the truncation.

By the inclusion of the nonlinear terms, the geometrical structure of the system becomes transparent. We show that the gauge transformation defined by the Lie 3-algebra can be identified as the diffeomorphism of $\mathcal{N}$ which preserve its volume 3-form. The gauge potential associated with this symmetry can be identified with two-form gauge field $b_{\mu\dot{\nu}}$ (an index μ for the world-volume and another $\dot{\nu}$ for the internal space $\mathcal{N}$) which is a particular combination of the Bagger-Lambert gauge field $A_{\mu ab}$. We show that only a particular combination of $A_{\mu ab}$ is relevant to define the gauge symmetry, the action and the supersymmetry. We note that the internal space $\mathcal{N}$ may be regarded as the fiber on the three dimensional membrane world-volume $\mathcal{M}$ in a sense.

The second characteristic feature of the system is that not only the covariant derivative defined by the gauge potential $b_{\mu\dot{\nu}}$ is covariant, the triplet commutator $\{X^{\dot{\mu}}, X^{\dot{\nu}}, \Phi\}$ is also covariant. This follows from the fundamental identity of the Nambu-Poisson structure. From this combination, we obtain the second two-form field $b_{\dot{\mu}\dot{\nu}}$ by which we can define the covariant derivative in the fiber direction $\mathcal{N}$. By combining two covariant derivatives, one obtains various six dimensional field strengths associated with $b_{\mu\dot{\nu}}, b_{\dot{\mu}\dot{\nu}}$.

The BLG action and the equations of motion are rewritten in terms of these fields. The equations of motion for the tensor field are written in a manifestly gauge-covariant form and combined with the Bianchi identity into a self-dual form.

We organize the paper as follows. In section 4, we derive the BLG gauge symmetry associated with the Nambu-Poisson bracket and identify the gauge fields $b_{\mu\dot{\nu}}, b_{\dot{\mu}\dot{\nu}}$ from the gauge field $A_{\mu ab}$ and the scalar field $X^{\dot{\mu}}$. In particular we identify the gauge transformation as the volume-preserving diffeomorphism of $\mathcal{N}$. In section 5 we derive two types of covariant derivatives from two-form gauge fields and the corresponding field strengths. The BLG action is then rewritten in terms of these fields and we derive the equation of motion. As mentioned, it is identical to the equation for self-dual field strength with the source terms associated with other fields. In section 6, we derive the supersymmetry transformation of the six dimensional fields.

In [8], the connection with M5 brane was used to provide the geometrical origin of extra generators in the construction of Lie 3-algebra which contains arbitrary Lie algebra. In section 7 we provide a detailed explanation of the derivation of D4 action from BLG model by the double dimensional reduction. Here the volume-preserving diffeomorphism is replaced by the area-preserving diffeomorphism. By comparing the obtained D4-brane action to the known result, we find explicit relations between parameters in the BLG model and those in M-theory (the Planck scale and the magnitude of the background C-field.) This clearly indicates that the BLG model well describes an M5-brane in a large C-field background.

In section 9, we give a few conjectural arguments which may be helpful to understand the geometrical nature of M5 brane in the future. First, in section 9 we point out that the M5-brane theory we obtained may be interpreted as a dynamical theory for the Nambu-Poisson structure. In this sense it is analogous to the Kodaira-Spencer theory [9] for the complex structure of a Calabi-Yau manifold.

JHEP08(2008)014

The last section is devoted to additional remarks and speculations.

For other recent developments of the BLG model, see [10].

2. Review of BLG model

Lie 3-algebra. The novelty of the BLG model is that it integrates a novel symmetry defined by Lie 3-algebra with supersymmetry. The Lie 3-algebra is defined by an antisymmetric trilinear product, called Nambu bracket, which will be represented by the bracket $\{*,*,*\}$. We denote the basis of the algebra be T^a. The consistency condition of Lie 3-algebra is that it must satisfy the so-called fundamental identity:

$$\begin{aligned}\{T^a,T^b,\{T^c,T^d,T^e\}\} &= \{\{T^a,T^b,T^c\},T^d,T^e\} \\ &\quad +\{T^c,\{T^a,T^b,T^d\},T^e\}+\{T^c,T^d,\{T^a,T^b,T^e\}\}. \end{aligned} \tag{2.1}$$

It is often convenient to define the structure constant ${f^{abc}}_d$ by

$$\{T^a,T^b,T^c\} = {f^{abc}}_d T^d. \tag{2.2}$$

For the construction of an action we need an invariant metric

$$h^{ab} = \langle T^a, T^b\rangle \tag{2.3}$$

which satisfies,

$$\langle\{T^a,T^b,T^c\},T^d\rangle + \langle T^c,\{T^a,T^b,T^d\}\rangle = 0\,. \tag{2.4}$$

With the Lie 3-algebra, various fields in BLG model which are symbolically written as $\phi = \sum_a \phi_a T^a$ transform infinitesimally as

$$\delta_\Lambda\phi = \sum_{a,b}\Lambda_{ab}\{T^a,T^b,\phi\},\quad \text{or}\quad \delta_\Lambda\phi_a = \Lambda_{cd}{f^{cdb}}_a\phi_b \tag{2.5}$$

for the gauge parameter Λ_{ab}. The fundamental identity implies that this transformation closes in the following sense,

$$[\delta_{\Lambda_1},\delta_{\Lambda_2}]\phi = \delta_{[\Lambda_1,\Lambda_2]}\phi,\quad [\Lambda_1,\Lambda_2]_{ab} := \Lambda_{1de}\Lambda_{2cb}{f^{dec}}_a + \Lambda_{1de}\Lambda_{2ac}{f^{dec}}_b\,. \tag{2.6}$$

As a result of (2.4), the metric (2.3) must also be invariant under the symmetry (2.5),

$$\langle\delta_\Lambda\phi_1,\phi_2\rangle + \langle\phi_1,\delta_\Lambda\phi_2\rangle = 0\,. \tag{2.7}$$

The BLG model, whose action is constructed with the structure constant and the invariant metric, is a gauge theory associated with this symmetry.

JHEP08(2008)014

Bagger-Lambert action. With our notation, the action of the BLG model is given by

$$S = S_X + S_\Psi + S_{\rm CS} + S_{\rm int} + S_{\rm pot}, \tag{2.8}$$

where

$$S_X = -\frac{1}{2}\int_{\mathcal{M}} d^3x \langle D_\mu X^I, D^\mu X^I\rangle, \tag{2.9}$$

$$S_{\rm pot} = -\frac{1}{12}\int_{\mathcal{M}} d^3x \langle \{X^I, X^J, X^K\}, \{X^I, X^J, X^K\}\rangle, \tag{2.10}$$

$$S_{\rm CS} = \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda}\left(\frac{1}{2} f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{1}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef}\right), \tag{2.11}$$

$$S_\Psi = \frac{i}{2}\int_{\mathcal{M}} d^3x \langle \overline{\Psi}, \Gamma^\mu D_\mu \Psi\rangle, \tag{2.12}$$

$$S_{\rm int} = \frac{i}{4}\int_{\mathcal{M}} d^3x \langle \overline{\Psi}, \Gamma_{IJ}\{X^I, X^J, \Psi\}\rangle, \tag{2.13}$$

where the covariant derivatives are

$$D_\mu X^I_a = \partial_\mu X^I_a - f^{bcd}{}_a A_{\mu bc} X^I_d, \quad D_\mu \Psi_a = \partial_\mu \Psi^I_a - f^{bcd}{}_a A_{\mu bc}\Psi_d. \tag{2.14}$$

We denote the world-volume of the membrane as $\mathcal{M}$ and its coordinate as x^μ ($\mu = 0, 1, 2$). The supersymmetry transformation parameter ϵ and the fermion Ψ belong to $\mathbf{8}_s$ and $\mathbf{8}_c$ representations, respectively, of the SO(8) R-symmetry, and are represented as 32 component spinors satisfying

$$\Gamma^{\mu\nu\rho}\epsilon = +\epsilon^{\mu\nu\rho}\epsilon, \quad \Gamma^{\mu\nu\rho}\psi = -\epsilon^{\mu\nu\rho}\psi. \tag{2.15}$$

This Lagrangian has a gauge symmetry associated with the 3-algebra,

$$\delta_\Lambda X^I_a = f^{bcd}{}_a \Lambda_{bc} X^I_d, \quad \delta_\Lambda \Psi_a = f^{bcd}{}_a \Lambda_{bc}\Psi_d, \quad \delta_\Lambda A_{\mu ab} = D_\mu \Lambda_{ab}, \tag{2.16}$$

where

$$D_\mu \Lambda_{ab} = \partial_\mu \Lambda_{ab} - f^{cde}{}_a A_{\mu cd}\Lambda_{eb} + f^{cde}{}_b A_{\mu cd}\Lambda_{ea}. \tag{2.17}$$

The Bagger-Lambert action has the maximal ($\mathcal{N} = 8$) SUSY in $d = 3$,

$$\delta X^I = i\bar{\epsilon}\Gamma^I \Psi, \tag{2.18}$$

$$\delta\Psi = D_\mu X^I \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6}\left\{X^I, X^J, X^K\right\}\Gamma^{IJK}\epsilon, \tag{2.19}$$

$$\delta \tilde{A}_\mu{}^b{}_a = i\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cdb}{}_a, \qquad \tilde{A}_\mu{}^b{}_a := f^{cdb}{}_a A_{\mu cd}. \tag{2.20}$$

3. Nambu-Poisson bracket and promotion of 3d fields to 6d

Nambu-Poisson bracket as Lie 3-algebra. For the construction of M5-brane, we introduce an "internal" three-manifold $\mathcal{N}$ and use the Nambu-Poisson bracket

$$\{f, g, h\}_{\rm NP} = \sum_{\dot\mu\dot\nu\dot\lambda} P^{\dot\mu\dot\nu\dot\lambda}(y)\partial_{\dot\mu} f \partial_{\dot\nu} g \partial_{\dot\lambda} h \tag{3.1}$$

JHEP08(2008)014

on $\mathcal{N}$ as a realization of three-algebra. Here $y^{\dot{\mu}}$ ($\dot{\mu} = \dot{1}, \dot{2}, \dot{3}$) is the local coordinate on $\mathcal{N}$. For literatures on the Nambu-Poisson bracket, see for example [11, 12]. One of the most important properties of the Nambu-Poisson bracket is that it satisfy the analog of the fundamental identity for arbitrary functions f_i ($i = 1, \ldots, 5$) on $\mathcal{N}$,

$$\begin{aligned}\{f_1, f_2, \{f_3, f_4, f_5\}_{\mathrm{NP}}\}_{\mathrm{NP}} &= \{\{f_1, f_2, f_3\}_{\mathrm{NP}}, f_4, f_5\}_{\mathrm{NP}} \\ &\quad + \{f_3, \{f_1, f_2, f_4\}_{\mathrm{NP}}, f_5\}_{\mathrm{NP}} + \{f_3, f_4, \{f_1, f_2, f_3\}_{\mathrm{NP}}\}_{\mathrm{NP}}. \end{aligned} \tag{3.2}$$

This gives a very severe constraint on the coefficient $P^{\dot{\mu}\dot{\nu}\dot{\lambda}}(y)$. Actually it is known that by the suitable choice of the local coordinates, it can be reduced to the Jacobian,

$$\{f, g, h\}_{\mathrm{NP}} = \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}} \frac{\partial f}{\partial y^{\dot{\mu}}} \frac{\partial g}{\partial y^{\dot{\nu}}} \frac{\partial h}{\partial y^{\dot{\rho}}}. \tag{3.3}$$

This property is referred to as the "decomposability" in the literature [12]. By using this fact, we can use (3.3) in the following without losing generality. We also note that the dimension of the internal manifold $\mathcal{N}$ is essentially restricted to 3 because of the decomposability. If we choose the basis of functions on $\mathcal{N}$ as $\chi^a(y)$ ($a = 1, 2, 3, \ldots$) and write the Nambu-Poisson bracket as a Lie 3-algebra,

$$\{\chi^a, \chi^b, \chi^c\}_{\mathrm{NP}} = \sum_d f^{abc}{}_d \chi^d , \tag{3.4}$$

eq. (3.2) implies that the structure constant $f^{abc}{}_d$ here satisfies the fundamental identity.

The integration over the y-space can be used to define the invariant metric,

$$\langle f, g \rangle = \frac{1}{g^2} \int_{\mathcal{N}} d^3y f(y) g(y). \tag{3.5}$$

It is obvious that this satisfies (2.4). We define

$$h^{ab} = \langle \chi^a, \chi^b \rangle, \quad h_{ab} = (h^{-1})_{ab} . \tag{3.6}$$

Because we have already fixed the scale of $y^{\dot{\mu}}$ at (3.3), we cannot in general remove the coefficient g from the metric (3.6). As we will show later, however, if the internal space is $\mathcal{N} = \mathbf{R}^3$, it is possible to set this coupling at an arbitrary value by an appropreate re-scaling of variables.

Except for the trivial case ($\mathcal{N} = \mathbf{R}^3$), we have to cover $\mathcal{N}$ by local patches and the coordinates $y^{\dot{\mu}}$ are the local coordinates on each patch. If we need to go to the different patch where the local coordinates are y', the coordinate transformation between y and y' (say $y'^{\dot{\mu}} = f^{\dot{\mu}}(y)$) should keep the Nambu-Poisson bracket (3.3). It implies that

$$\{f^{\dot{1}}, f^{\dot{2}}, f^{\dot{3}}\} = 1 . \tag{3.7}$$

Namely $f^{\dot{\mu}}(y)$ should be the volume-preserving diffeomorphism. As we will see, the gauge symmetry of the BLG model for this choice of Lie 3-algebra is the volume-preserving diffeomorphism of $\mathcal{N}$ which is very natural in this set-up.

We note that we do not need the metric in $y^{\dot{\mu}}$ space. For the definition of the theory we only need to specify a volume form in $\mathcal{N}$. The gauge symmetry associated with the volume-preserving diffeomorphism is kept not by the metric but the various components of the self-dual two-form field which comes out from $A_{\mu ab}$ and $X^{\dot{\mu}}$ (longitudinal components of X) as we will see.

JHEP08(2008)014

Definition of 6 dim fields. By combining the basis of $C(\mathcal{N})$, we can treat $X^I_a(x)$ and $\Psi_a(x)$ as six-dimensional local fields

$$X^I(x,y) = \sum_a X^I_a(x)\chi^a(y), \quad \Psi(x,y) = \sum_a \Psi_a(x)\chi^a(y). \tag{3.8}$$

Similarly, the gauge field A^{ab}_λ can be regarded as a bi-local field:

$$A_\lambda(x,y,y') = A^{ab}_\lambda(x)\chi^a(y)\chi^b(y'). \tag{3.9}$$

The existence of such a bi-local field does not mean the theory is non-local. Let us expand it with respect to $\Delta y^{\dot\mu} \equiv y'^{\dot\mu} - y^{\dot\mu}$ as

$$A_\lambda(x,y,y') = a_\lambda(x,y) + b_{\lambda\dot\mu}(x,y)\Delta y^{\dot\mu} + \frac{1}{2}c_{\lambda\dot\mu\dot\nu}(x,y)\Delta y^{\dot\mu}\Delta y^{\dot\nu} + \cdots . \tag{3.10}$$

Because $A_{\lambda ab}$ always appears in the action in the form $f^{bcd}{}_a A_{\lambda bc}$, the field $A_\lambda(y,y')$ is highly redundant, and only the component

$$b_{\lambda\dot\mu}(x,y) = \left.\frac{\partial}{\partial y'^{\dot\mu}} A_\lambda(x,y,y')\right|_{y'=y} \tag{3.11}$$

contributes to the action.[1] For example, the covariant derivative (2.14) of BLG model is rewritten for our case as,

$$\begin{aligned} D_\lambda X^I(x,y) &\equiv (\partial_\lambda X^{Ia}(x) - g f^{bcd}{}_a A_{\lambda bc} X^I_d(x))\chi^a(y) \\ &= \partial_\lambda X^I(x,y) - g\epsilon^{\dot\mu\dot\nu\dot\rho} \left.\frac{\partial^2 A_\lambda(x,y,y')}{\partial y^{\dot\mu}\partial y'^{\dot\nu}}\right|_{y=y'} \frac{\partial X^I(x,y)}{\partial y^{\dot\rho}} \\ &= \partial_\lambda X^I(x,y) - g\epsilon^{\dot\mu\dot\nu\dot\rho}(\partial_{\dot\mu} b_{\lambda\dot\nu}(x,y))(\partial_{\dot\rho} X^I(x,y)) \\ &= \partial_\lambda X^I - g\{b_{\lambda\dot\nu}, y^{\dot\nu}, X^I\}. \end{aligned} \tag{3.12}$$

The covariant derivative for the fermion field is similarly,

$$D_\lambda \Psi(x,y) = \partial_\lambda \Psi(x,y) - g\epsilon^{\dot\mu\dot\nu\dot\rho}(\partial_{\dot\mu} b_{\lambda\dot\nu}(x,y))(\partial_{\dot\rho}\Psi(x,y)) = \partial_\lambda \Psi - g\{b_{\lambda\dot\nu}, y^{\dot\nu}, \Psi\}\,. \tag{3.13}$$

Longitudinal fields. In [1], this theory written in terms of fields on six dimensions is identified with the theory describing a single M5-brane. At this point, only the x^μ part of the metric $g_{\mu\nu} = \eta_{\mu\nu}$ is defined, and we still have SO(8) global symmetry, which is different from the SO(5) symmetry expected in the M5-brane theory.

This is quite similar to the situation in which we consider the D-brane Born-Infeld action. The Born-Infeld Dp-brane action of ten-dimensional superstring theory possesses SO$(1,9)$ Lorentz symmetry regardress of the world-volume dimension $p+1$. The rotational symmetry is reduced to SO$(9-p)$ for the transverse directions only after fixing the general coordinate transformation symmetry on the world-volume with the static gauge condition[2]

$$X^\mu(\sigma) = \sigma^\mu. \tag{3.14}$$

[1] In [1] it was treated as a trick (or an approximation by neglecting the irrelevant parts) to derive M5 action. However, it turned out that this is actually the exact statement.

[2] Turning on a background field such as the B-field will of course also break the global symmetry. For the discussion here we are treating the background fields as covariant dynamical fields.

This gauge fixing breaks the global symmetry from SO(1, 9) to SO(9 − p), and at the same time the world-volume metric is induced from the target space metric through (3.14).

We can interpret the six-dimensional theory we are considering here as a theory obtained from an SO(1, 10) symmetric covariant theory by taking a partial static gauge for three among six world-volume coordinates. As we mentioned above, however, we do not have full diffeomorphism in the $y^{\dot{\mu}}$ space. The action is invariant only under volume-preserving diffeomorphism. This implies that we cannot completely fix the fields $X^{\dot{\mu}}$, and there are remaining physical degrees of freedom. For this reason, we should loosen the static gauge condition as [1]

$$X^{\dot{\mu}}(x, y) = y^{\dot{\mu}} + b^{\dot{\mu}}(x, y), \quad b_{\dot{\mu}\dot{\nu}} = \epsilon_{\dot{\mu}\dot{\nu}\dot{\rho}} b^{\dot{\rho}}. \tag{3.15}$$

As was shown in [1], the tensor field $b_{\dot{\mu}\dot{\nu}}$ is identified with a part of the 2-form gauge field on a M5-brane.

Comments on the coupling constant. In the case of ordinary Yang-Mills theories, there are two widely-used conventions for coupling constants and normalization of gauge fields. One way is to normalize a gauge field by the canonical kinetic term $-(1/4)F_{\mu\nu}^2$ and put the coupling constant in the covariant derivative $D = d - igA$. The other choice is to define the covariant derivative $D = d - iA$ without using the coupling constant and instead put $1/g^2$ in front of the kinetic term of the gauge field. Similarly, there are different conventions for coupling constant in the case of the BL theory, too. In the above, we put the coupling constant g in the definition of the metric (3.6). This corresponds to the second convention we mentioned above. We can move the coupling dependence from the overall factor to the interaction terms by re-scaling the fields

$$X^I \to cX^I, \quad \Psi \to c\Psi, \quad b_{\mu\dot{\mu}} \to cb_{\mu\dot{\mu}}, \tag{3.16}$$

with $c = g$. In general, as ordinary Yang-Mills theories, we cannot remove the coupling constant completely from the action.

If the internal space $\mathcal{N}$ is $\mathbf{R}^3$, however, we have an extra degree of freedom for re-scaling, and it is in fact possible to the coupling constant from the action. Let us consider the following re-scaling of variables.

$$X^I \to c'^3 X^I, \quad \Psi \to c'^3\Psi, \quad b_{\mu\dot{\mu}} \to c'^4 b_{\mu\dot{\mu}}, \quad y^{\dot{\mu}} \to c'^2 y^{\dot{\mu}}. \tag{3.17}$$

This variable change is associated with an outer automorphism of the algebra, and does not change the relative coefficients in the action. The only change in the action is the overall factor. We can thus absorb the coupling constant by (3.17), and this implies that the six-dimensional theory does not have any coupling constant.

We can adopt an elegant convention in which no coupling constant appears. However, we adopt a different convention below. Because we interpret the six-dimensional theory as a theory of an M5-brane, we would like to regard the scalar field X^I as the coordinates of the target space with mass dimension −1. We also give the meaning to the variables $y^{\dot{\mu}}$ as the world-volume coordinates, which also have mass dimension −1. We choose the

JHEP08(2008)014

parametrization in the $y^{\dot\mu}$ space so that the linear part of the six-dimensional action is invariant under Lorentz transformations in the $(x^\mu, y^{\dot\mu})$ space. After fixing the scale of X^I and $y^{\dot\mu}$ in this way, we can no longer use the two re-scalings (3.16) and (3.17) to change the coupling constant and overall coefficient of the action. These two parameters have physical meaning now.

In the following, in order to express the coupling constant dependence of each term in the action clearly, we separate the coupling constant g from the structure constant. We also introduce an overall coefficient T_6, which is regarded as an effective tension of the M5-brane. This plays an important role in the parameter matching in section 7, but we will omit this factor in section 4, section 5, and section 6 for simplicity because it is irrelevant to the analysis in these sections.

4. Gauge symmetry of M5 from Lie 3-algebra

Gauge transformation. The gauge transformations of the scalar fields X^I and fermion fields Ψ are given by

$$\begin{aligned} \delta_\Lambda X^I(x,y) &= g\Lambda_{ab}(x) f^{abc}{}_d X^I_c(x)\chi^d(y) \\ &= g\Lambda_{ab}(x)\{\chi^a,\chi^b,X^I\} = g(\delta_\Lambda y^{\dot\rho})\partial_{\dot\rho}X^I(x,y), \\ \delta_\Lambda \Psi(x,y) &= g\Lambda_{ab}(x)\{\chi^a,\chi^b,\Psi\} = g(\delta_\Lambda y^{\dot\rho})\partial_{\dot\rho}\Psi(x,y), \end{aligned} \tag{4.1}$$

where we used

$$f^{abc}{}_d = \langle\{\chi^a,\chi^b,\chi^c\},\chi_d\rangle\,, \quad \sum_a \chi^a(y)\chi_a(y') = \delta^{(3)}(y-y')\,. \tag{4.2}$$

$\delta_\Lambda y^{\dot\mu}$ is defined as

$$\delta_\Lambda y^{\dot\lambda} = \epsilon^{\dot\lambda\dot\mu\dot\nu}\partial_{\dot\mu}\Lambda_{\dot\nu}(x,y), \tag{4.3}$$

$$\Lambda_{\dot\mu}(x,y) = \partial'_{\dot\mu}\tilde\Lambda(x,y,y')|_{y'=y}\,, \quad \tilde\Lambda(x,y,y') := \Lambda_{ab}(x)\chi^a(y)\chi^b(y'). \tag{4.4}$$

We note that although the parameter of a gauge transformation may be expressed as a bi-local function $\tilde\Lambda(x,y,y')$, the gauge transformation induced by it depends only on its component $\Lambda_{\dot\mu}(x,y)$ which is local in $\mathcal{N}$. It comes from the fact that the gauge transformation by Λ_{ab} is always defined through the combination $f^{abc}{}_d\Lambda_{ab}$.

The same argument can be applied to the gauge field $A_\mu(x,y,y')$. As we already mentioned, since it appears only through the combination $A_{\mu ab}f^{abc}{}_d$, the local field $b_{\mu\dot\lambda}(x,y)$ defined as (3.11) shows up in the action.

The transformation (4.1) may be regarded as the infinitesimal reparametrization

$$y'^{\dot\lambda} = y^{\dot\lambda} - g\delta y^{\dot\lambda}. \tag{4.5}$$

Since $\partial_{\dot\mu}\delta y^{\dot\mu} = 0$, it represents the volume-preserving diffeomorphism. Since the symmetry is local on $\mathcal{M}$, the gauge parameter is an arbitrary function of x. So what we have obtained is a gauge theory on $\mathcal{M}$ whose gauge group is the volume-preserving diffeomorphism of $\mathcal{N}$.

In this sense, the world-volume of M5 brane may be regarded as the vector bundle $\mathcal{N} \to \mathcal{M}$ but the gauge transformation on each fiber is not merely the linear transformation but the diffeomorphism on the fiber which preserves the volume form

$$\omega = dy^{\dot{1}} \wedge dy^{\dot{2}} \wedge dy^{\dot{3}}. \tag{4.6}$$

As we mentioned in the previous section, among eight scalar fields X^I, the last five components X^i are treated as scalar fields representing the transverse fluctuations of the M5-brane. The other three $X^{\dot{\mu}}$ (longitudinal field) are rewritten as as

$$X^{\dot{\mu}}(y) = \frac{y^{\dot{\mu}}}{g} + \frac{1}{2}\epsilon^{\dot{\mu}\dot{\kappa}\dot{\lambda}} b_{\dot{\kappa}\dot{\lambda}}(y). \tag{4.7}$$

We chose the coefficients so that we obtain Lorentz invariant kinetic terms in the six-dimensional action. The gauge transformation of $b_{\dot{\mu}\dot{\nu}}$ can be derived from (4.1) and (4.7) as

$$\delta_\Lambda b_{\dot{\kappa}\dot{\lambda}}(y) = \partial_{\dot{\kappa}}\Lambda_{\dot{\lambda}} - \partial_{\dot{\lambda}}\Lambda_{\dot{\kappa}} + g(\delta_\Lambda y^{\dot{\rho}})\partial_{\dot{\rho}} b_{\dot{\kappa}\dot{\lambda}}(y). \tag{4.8}$$

The gauge transformation of the gauge field $A_\lambda(x, y, y')$ is given by $\delta_\Lambda A_\lambda(x, y, y') = D_\lambda \tilde{\Lambda}(x, y, y')$. The covariant derivative of a bi-local field is defined by tensoring the covariant derivative (3.12) for a local field, and we obtain

$$D_\lambda \Lambda(y, y') = \partial_\lambda \Lambda(y, y') - g\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}[\partial_{\dot{\mu}} b_{\lambda\dot{\nu}}(y)\partial_{\dot{\rho}}\Lambda(y, y') + \partial'_{\dot{\mu}} b_{\lambda\dot{\nu}}(y')\partial'_{\dot{\rho}}\Lambda(y, y')]. \tag{4.9}$$

From this we can extract the transformation law of the component field $b_{\lambda\dot{\sigma}}$

$$\delta_\Lambda b_{\lambda\dot{\sigma}} = \partial'_{\dot{\mu}}\delta_\Lambda A_\lambda(y, y')|_{y'=y} = \partial_\lambda \Lambda_{\dot{\sigma}} - g\partial_{\dot{\sigma}}\xi_\Lambda - g\delta_{\mathrm{gc}} b_{\lambda\dot{\sigma}}, \tag{4.10}$$

where $\delta_{\mathrm{gc}} b_{\lambda\dot{\sigma}}$ is the coordinate transformation in y-space

$$\delta_{\mathrm{gc}} b_{\lambda\dot{\sigma}} = -\delta_\Lambda y^{\dot{\tau}}\partial_{\dot{\tau}} b_{\lambda\dot{\sigma}} - (\partial_{\dot{\sigma}}\delta_\Lambda y^{\dot{\tau}}) b_{\lambda\dot{\tau}}, \tag{4.11}$$

and ξ_Λ is defined by

$$\xi_\Lambda = \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}(\partial_{\dot{\mu}} b_{\lambda\dot{\nu}}\Lambda_{\dot{\rho}} + b_{\lambda\dot{\mu}}\partial_{\dot{\nu}}\Lambda_{\dot{\rho}}). \tag{4.12}$$

In addition to these gauge transformations derived from (2.14) and (2.17), there is an additional gauge transformation which acts only on the field $b_{\lambda\dot{\mu}}$. As we can see in (3.12), $b_{\lambda\dot{\mu}}$ appears in the covariant derivative in the form of the rotation in the $y^{\dot{\mu}}$ space. This means that $D_\mu\Phi$ is invariant under

$$\delta b_{\lambda\dot{\mu}} = -\partial_{\dot{\mu}}\Lambda_\lambda. \tag{4.13}$$

We can easily check that the Chern-Simons term is also invariant under this transformation, and thus (4.13) is also a gauge symmetry of the theory.

Now we summarize the gauge transformation of the six-dimensional theory.

$$\delta_\Lambda X^i = g(\delta_\Lambda y^{\dot{\rho}})\partial_{\dot{\rho}} X^i, \tag{4.14}$$

$$\delta_\Lambda \Psi = g(\delta_\Lambda y^{\dot{\rho}})\partial_{\dot{\rho}}\Psi, \tag{4.15}$$

$$\delta_\Lambda b_{\dot{\kappa}\dot{\lambda}} = \partial_{\dot{\kappa}}\Lambda_{\dot{\lambda}} - \partial_{\dot{\lambda}}\Lambda_{\dot{\kappa}} + g(\delta_\Lambda y^{\dot{\rho}})\partial_{\dot{\rho}} b_{\dot{\kappa}\dot{\lambda}}, \tag{4.16}$$

$$\delta_\Lambda b_{\lambda\dot{\sigma}} = \partial_\lambda\Lambda_{\dot{\sigma}} - \partial_{\dot{\sigma}}\Lambda_\lambda - g\delta_{\mathrm{gc}} b_{\lambda\dot{\sigma}}. \tag{4.17}$$

We absorbed ξ_Λ in (4.10) into the definition of the parameter Λ_λ. In the weak coupling limit $g \to 0$, we obtain the standard gauge transformation on an M5-brane.

JHEP08(2008)014

Covariant derivatives in 6 dim. An intriguing feature of our six dimensional model is that one may define the covariant derivative in the *fiber* direction.

By using the fundamental identity, it is easy to show that if Φ_1, Φ_2, and Φ_3 are covariant fields (such as X^I or Ψ), not only $D_\mu \Phi_1$ but $\{\Phi_1, \Phi_2, \Phi_3\}$ are also covariant because of the fundamental identity,

$$\delta_\Lambda \{\Phi_1, \Phi_2, \Phi_3\} = \{\delta_\Lambda \Phi_1, \Phi_2, \Phi_3\} + \{\Phi_1, \delta_\Lambda \Phi_2, \Phi_3\} + \{\Phi_1, \Phi_2, \delta_\Lambda \Phi_3\}\,. \tag{4.18}$$

It implies that the following combination defines the "covariant" derivative along the fiber direction,

$$\begin{aligned}\mathcal{D}_{\dot\mu}\Phi &\equiv \frac{g^2}{2}\epsilon_{\dot\mu\dot\nu\dot\rho}\{X^{\dot\nu}, X^{\dot\rho}, \Phi\} \\ &= \partial_{\dot\mu}\Phi + g(\partial_{\dot\lambda} b^{\dot\lambda}\partial_{\dot\mu}\Phi - \partial_{\dot\mu} b^{\dot\lambda}\partial_{\dot\lambda}\Phi) + \frac{g^2}{2}\epsilon_{\dot\mu\dot\nu\dot\rho}\{b^{\dot\nu}, b^{\dot\rho}, \Phi\}.\end{aligned} \tag{4.19}$$

Together with (3.12), which we repeat here again,

$$\mathcal{D}_\mu \Phi \equiv D_\mu \Phi = \partial_\mu \Phi - g\{b_{\mu\dot\nu}, y^{\dot\nu}, \Phi\}, \tag{4.20}$$

we have a set of covariant derivatives on M5 world-volume.

These covariant derivatives possess the following important properties.

- Leibniz rule:

$$\mathcal{D}_{\underline{\mu}}\{\Phi_1, \Phi_2, \Phi_3\} = \{\mathcal{D}_{\underline{\mu}}\Phi_1, \Phi_2, \Phi_3\} + \{\Phi_1, \mathcal{D}_{\underline{\mu}}\Phi_2, \Phi_3\} + \{\Phi_1, \Phi_2, \mathcal{D}_{\underline{\mu}}\Phi_3\}. \tag{4.21}$$

- Integration by parts:

$$\int d^3x d^3y \Phi_1 \mathcal{D}_{\underline{\mu}}\Phi_2 = -\int d^3x d^3y (\mathcal{D}_{\underline{\mu}}\Phi_1)\Phi_2. \tag{4.22}$$

Here $\mathcal{D}_{\underline{\mu}}$ ($\underline{\mu} = 0, 1, \ldots, 5$) represents $\mathcal{D}_\mu$ and $\mathcal{D}_{\dot\mu}$.

Field strength. As special cases of these covariant derivatives, we define the following field strengths of the tensor field.

$$\begin{aligned}\mathcal{H}_{\lambda\dot\mu\dot\nu} &= \epsilon_{\dot\mu\dot\nu\dot\lambda}\mathcal{D}_\lambda X^{\dot\lambda} \\ &= H_{\lambda\dot\mu\dot\nu} - g\epsilon^{\dot\sigma\dot\tau\dot\rho}(\partial_{\dot\sigma} b_{\lambda\dot\tau})\partial_{\dot\rho} b_{\dot\mu\dot\nu},\end{aligned} \tag{4.23}$$

$$\begin{aligned}\mathcal{H}_{\dot1\dot2\dot3} &= g^2\{X^{\dot1}, X^{\dot2}, X^{\dot3}\} - \frac{1}{g} = \frac{1}{g}(V-1) \\ &= H_{\dot1\dot2\dot3} + \frac{g}{2}(\partial_{\dot\mu} b^{\dot\mu}\partial_{\dot\nu} b^{\dot\nu} - \partial_{\dot\mu} b^{\dot\nu}\partial_{\dot\nu} b^{\dot\mu}) + g^2\{b^{\dot1}, b^{\dot2}, b^{\dot3}\},\end{aligned} \tag{4.24}$$

where V is the "induced volume"

$$V = g^3\{X^{\dot1}, X^{\dot2}, X^{\dot3}\}, \tag{4.25}$$

JHEP08(2008)014

and H is the linear part of the field strength

$$H_{\lambda\dot{\mu}\dot{\nu}} = \partial_\lambda b_{\dot{\mu}\dot{\nu}} - \partial_{\dot{\mu}} b_{\lambda\dot{\nu}} + \partial_{\dot{\nu}} b_{\lambda\dot{\mu}}, \tag{4.26}$$

$$H_{\dot{\lambda}\dot{\mu}\dot{\nu}} = \partial_{\dot{\lambda}} b_{\dot{\mu}\dot{\nu}} + \partial_{\dot{\mu}} b_{\dot{\nu}\dot{\lambda}} + \partial_{\dot{\nu}} b_{\dot{\lambda}\dot{\mu}}. \tag{4.27}$$

$\mathcal{H}$ are covariantly transformed under the gauge transformation.

Just like the case of ordinary gauge theories, the field strength $\mathcal{H}$ arises in the commutator of the covariant derivatives defined above:

$$[\mathcal{D}_{\dot{\mu}}, \mathcal{D}_{\dot{\nu}}]\Phi = g^2 \epsilon_{\dot{\nu}\dot{\mu}\dot{\sigma}} \{\mathcal{H}_{\dot{1}\dot{2}\dot{3}}, X^{\dot{\sigma}}, \Phi\}, \tag{4.28}$$

$$[\mathcal{D}_\lambda, \mathcal{D}_{\dot{\lambda}}]\Phi = g^2 \{\mathcal{H}_{\lambda\dot{\nu}\dot{\lambda}}, X^{\dot{\nu}}, \Phi\}, \tag{4.29}$$

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\Phi = -\frac{g}{V} \epsilon_{\mu\nu\lambda} \mathcal{D}_\rho \widetilde{\mathcal{H}}^{\rho\lambda\dot{\kappa}} \mathcal{D}_{\dot{\kappa}} \Phi, \tag{4.30}$$

where the dual field strength $\widetilde{\mathcal{H}}$ is defined by

$$\widetilde{\mathcal{H}}^{\lambda\rho\dot{\kappa}} = \frac{1}{2} \epsilon^{\lambda\rho\dot{\kappa}\sigma\dot{\mu}\dot{\nu}} \mathcal{H}_{\sigma\dot{\mu}\dot{\nu}}, \quad \widetilde{\mathcal{H}}^{\mu\nu\rho} = \frac{1}{6} \epsilon^{\mu\nu\rho\dot{\mu}\dot{\nu}\dot{\rho}} \mathcal{H}_{\dot{\mu}\dot{\nu}\dot{\rho}}. \tag{4.31}$$

5. M5 action and equation of motion

We rewrite the various parts of the Bagger-Lambert action in terms of the six dimensional fields and their covariant derivatives,

$$\begin{aligned} S_X + S_{\text{pot}} &= \int d^3x \left\langle -\frac{1}{2}(\mathcal{D}_\mu X^i)^2 - \frac{1}{2}(\mathcal{D}_{\dot{\lambda}} X^i)^2 - \frac{1}{4}\mathcal{H}^2_{\lambda\dot{\mu}\dot{\nu}} - \frac{1}{12}\mathcal{H}^2_{\dot{\mu}\dot{\nu}\dot{\rho}} \right. \\ &\qquad \left. -\frac{1}{2g^2} - \frac{g^4}{4}\{X^{\dot{\mu}}, X^i, X^j\}^2 - \frac{g^4}{12}\{X^i, X^j, X^k\}^2 \right\rangle, \end{aligned} \tag{5.1}$$

$$\begin{aligned} S_\Psi + S_{\text{int}} &= \int d^3x \left\langle \frac{i}{2}\overline{\Psi}\Gamma^\mu \mathcal{D}_\mu \Psi + \frac{i}{2}\overline{\Psi}\Gamma^{\dot{\rho}}\Gamma_{\dot{1}\dot{2}\dot{3}} \mathcal{D}_{\dot{\rho}} \Psi \right. \\ &\qquad \left. + \frac{ig^2}{2}\overline{\Psi}\Gamma_{\dot{\mu}i}\{X^{\dot{\mu}}, X^i, \Psi\} + \frac{ig^2}{4}\overline{\Psi}\Gamma_{ij}\{X^i, X^j, \Psi\} \right\rangle. \end{aligned} \tag{5.2}$$

The scalar kinetic term is manifestly Lorentz symmetric up to the different structure inside the covariant derivatives $\mathcal{D}_\mu$ and $\mathcal{D}_{\dot{\mu}}$. The Chern-Simons term cannot be rewritten in manifestly gauge-covariant form.

$$\begin{aligned} S_{\text{CS}} &= \int d^3x \epsilon^{\mu\nu\lambda} \left\langle -\frac{1}{2}\epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\mu}} b_{\mu\dot{\nu}} \partial_\nu b_{\lambda\dot{\lambda}} \right. \\ &\qquad \left. + \frac{g}{6}\epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\mu}} b_{\nu\dot{\nu}} \epsilon^{\dot{\rho}\dot{\sigma}\dot{\tau}} \partial_{\dot{\sigma}} b_{\lambda\dot{\rho}} (\partial_{\dot{\lambda}} b_{\mu\dot{\tau}} - \partial_{\dot{\tau}} b_{\mu\dot{\lambda}}) \right\rangle \\ &= \int d^3x \int_y \epsilon^{\mu\nu\lambda} \left(-\frac{1}{2} db_\mu \wedge \partial_\nu b_\lambda - \frac{g}{6}(*db_\mu) \wedge (*db_\nu) \wedge (*db_\lambda) \right). \end{aligned} \tag{5.3}$$

In the second expression we treat $b_{\mu\dot{\mu}}$ as a one-form field $b_\mu = b_{\mu\dot{\mu}} dy^{\dot{\mu}}$ in the y-space. However, the equation of motion which is derived from these actions turns out to be manifestly gauge-covariant.

JHEP08(2008)014

Comments on fermion action. In the fermion kinetic terms in (5.2), only the $\mathrm{SO}(1,2) \times \mathrm{SO}(3)$ subgroup of the Lorentz symmetry is manifest due to the existence of $\Gamma_{\dot{1}\dot{2}\dot{3}}$ in one of two terms. We can remove this unwanted factor from the kinetic term by the unitary transformation

$$\overline{\Psi} = \overline{\Psi}' U, \quad \Psi = U\Psi', \tag{5.4}$$

where U is the matrix

$$U = \exp\left(-\frac{\pi}{4}\Gamma_{\dot{1}\dot{2}\dot{3}}\right) = \frac{1}{\sqrt{2}}(1 - \Gamma_{\dot{1}\dot{2}\dot{3}}). \tag{5.5}$$

The SUSY parameter ϵ is also transformed in the same way. Note that both Ψ and $\overline{\Psi}$ are transformed by U. This is consistent with the Dirac conjugation. As the result of the unitary transformation, the fermion terms in the action become

$$\begin{aligned} S_\Psi + S_{\text{int}} = \int d^3x \Big\langle & \frac{i}{2}\overline{\Psi}'\Gamma^\mu \mathcal{D}_\mu \Psi' + \frac{i}{2}\overline{\Psi}'\Gamma^{\dot{\rho}}\mathcal{D}_{\dot{\rho}}\Psi' \\ & + \frac{ig^2}{2}\overline{\Psi}'\Gamma_{\dot{\mu}i}\{X^{\dot{\mu}}, X^i, \Psi'\} - \frac{ig^2}{4}\overline{\Psi}'\Gamma_{ij}\Gamma_{\dot{1}\dot{2}\dot{3}}\{X^i, X^j, \Psi'\}\Big\rangle . \end{aligned} \tag{5.6}$$

After the unitary transformation, the condition (2.15) becomes the chirality condition in six dimension,

$$\Gamma^7\epsilon' = \epsilon', \quad \Gamma^7\Psi' = -\Psi', \tag{5.7}$$

where the chirality matrix Γ^7 is defined by

$$\Gamma^{\mu\nu\rho}\Gamma^{\dot{1}\dot{2}\dot{3}} = \epsilon^{\mu\nu\rho}\Gamma^7. \tag{5.8}$$

This means that the supersymmetry realized in this theory is the chiral $\mathcal{N} = (2,0)$ supersymmetry, which is the same as the supersymmetry on an M5-brane.

Equations of motion. It is easy to obtain the equations of motion for the scalar fields and fermion field as

$$\begin{aligned} 0 = {} & \mathcal{D}_\mu^2 X^i + \mathcal{D}_{\dot{\mu}}^2 X^i \\ & + g^4\{X^{\dot{\mu}}, X^j, \{X^{\dot{\mu}}, X^j, X^i\}\} + \frac{g^4}{2}\{X^j, X^k, \{X^j, X^k, X^i\}\} \\ & + \frac{ig^2}{2}\{\overline{\Psi}'\Gamma_{\dot{\mu}i}, X^{\dot{\mu}}, \Psi'\} + \frac{ig^2}{2}\{\overline{\Psi}'\Gamma_{ij}\Gamma_{\dot{1}\dot{2}\dot{3}}, X^j, \Psi'\}, \end{aligned} \tag{5.9}$$

$$0 = \Gamma^\mu \mathcal{D}_\mu \Psi' + \Gamma^{\dot{\rho}}\mathcal{D}_{\dot{\rho}}\Psi' + g^2\Gamma_{\dot{\mu}i}\{X^{\dot{\mu}}, X^i, \Psi'\} - \frac{g^2}{2}\Gamma_{ij}\Gamma_{\dot{1}\dot{2}\dot{3}}\{X^i, X^j, \Psi'\}. \tag{5.10}$$

The equations of motion of gauge fields $b_{\mu\dot{\mu}}$ and $b_{\dot{\mu}\dot{\mu}}$, and the Bianchi identity are combined into the self-dual form:

$$\mathcal{D}_\lambda \mathcal{H}^{\lambda\dot{\mu}\dot{\nu}} + \mathcal{D}_{\dot{\lambda}}\mathcal{H}^{\dot{\lambda}\dot{\mu}\dot{\nu}} = gJ^{\dot{\mu}\dot{\nu}}, \tag{5.11}$$

$$\mathcal{D}_\lambda \widetilde{\mathcal{H}}^{\lambda\mu\dot{\nu}} + \mathcal{D}_{\dot{\lambda}}\mathcal{H}^{\dot{\lambda}\mu\dot{\nu}} = gJ^{\mu\dot{\nu}}, \tag{5.12}$$

$$\mathcal{D}_\lambda \widetilde{\mathcal{H}}^{\lambda\mu\nu} + \mathcal{D}_{\dot{\lambda}}\widetilde{\mathcal{H}}^{\dot{\lambda}\mu\nu} = 0. \tag{5.13}$$

The first two are equations of motion obtained from the action, while the last one is a Bianchi identity derived from the commutation relation (4.30). The currents on the right hand sides are given by

$$\begin{aligned} J^{\dot{\rho}\dot{\sigma}} &= g(\{X^i, \mathcal{D}_{\dot{\sigma}} X^i, X^{\dot{\rho}}\} - (\dot{\rho} \leftrightarrow \dot{\sigma})) - \frac{g^3}{2}\epsilon^{\dot{\rho}\dot{\sigma}\dot{\mu}}\{X^i, X^j, \{X^i, X^j, X^{\dot{\mu}}\}\} \\ &\quad + \frac{ig}{2}(\{\overline{\Psi}'\Gamma^{\dot{\sigma}}, X^{\dot{\rho}}, \Psi'\} - (\dot{\rho} \leftrightarrow \dot{\sigma})) + \frac{ig}{2}\epsilon^{\dot{\rho}\dot{\sigma}\dot{\mu}}\{\overline{\Psi}'\Gamma_{\dot{\mu}i}, X^i, \Psi'\}, \end{aligned} \quad (5.14)$$

$$J^{\mu\dot{\nu}} = g\{X^i, \mathcal{D}_\mu X^i, X^{\dot{\nu}}\} + \frac{ig}{2}\{\overline{\Psi}'\Gamma^\mu, \Psi', X^{\dot{\nu}}\}. \quad (5.15)$$

The self-dual tensor field $\mathcal{H}$, chiral fermion field Ψ', and the five scalar fields X^i form a tensor multiplet of $\mathcal{N} = (2,0)$ supersymmetry [13], which is the same as the field contents on an M5-brane.

6. Supersymmetry

In this section we rewrite the supersymmetry transformations (2.18)-(2.20) in terms of the six-dimensional covariant derivatives and field strength. The transformation law (2.20) of the gauge field $A_{\mu ab}$ with coupling constant inderted is

$$\widetilde{A}_\mu{}^b{}_a = ig\bar{\epsilon}\Gamma_\mu\Gamma_I X^I_c \Psi_d f^{cdb}{}_a. \quad (6.1)$$

We cannot determine uniquely the transformation law of the component field $b_{\mu\dot{\nu}}$ from this equation because of the existence of the gauge transformation (4.13), which acts only on $b_{\mu\dot{\mu}}$. In fact, the transformation (6.1) only gives

$$\delta\left(\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}\partial_{\dot{\mu}} b_{\lambda\dot{\nu}}\partial_{\dot{\rho}} f(y)\right) = ig\bar{\epsilon}\Gamma_\lambda\Gamma_I\{X^I, \Psi, f(y)\}, \quad (6.2)$$

where $f(y)$ is an arbitrary function of $y^{\dot{\mu}}$. One possible choice for $\delta b_{\mu\dot{\mu}}$ is

$$\delta b_{\mu\dot{\nu}} = ig(\bar{\epsilon}\Gamma_I\Gamma_\mu\Psi)\partial_{\dot{\nu}} X^I. \quad (6.3)$$

We can easily check that this transformation law reproduces (6.2).

In some situations an explicit appearance of $b_{\mu\dot{\mu}}$ is not necessary, but all we need is $B_\mu{}^{\dot{\mu}} \equiv \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}\partial_{\dot{\nu}} b_{\mu\dot{\rho}}$, which satisfies the constraint $\partial_{\dot{\mu}} B_\mu{}^{\dot{\mu}} = 0$. The SUSY transformation for $B_\mu{}^{\dot{\mu}}$ is uniquely determined from (6.2) as

$$\delta B_\mu{}^{\dot{\mu}} = ig\bar{\epsilon}\Gamma_\mu\Gamma_I\epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}}\partial_{\dot{\nu}} X^I \partial_{\dot{\lambda}}\Psi, \quad (6.4)$$

and it is obvious that the constraint is SUSY invariant, i.e.

$$\delta(\partial_{\dot{\mu}} B_\mu{}^{\dot{\mu}}) = 0. \quad (6.5)$$

The transformation laws rewritten in terms of the six-dimensional notation are

$$\delta X^i = i\overline{\epsilon}'\Gamma^i\Psi', \quad (6.6)$$

$$\begin{aligned} \delta\Psi' &= \mathcal{D}_\mu X^i\Gamma^\mu\Gamma^i\epsilon' + \mathcal{D}_{\dot{\mu}} X^i\Gamma^{\dot{\mu}}\Gamma^i\epsilon' \\ &\quad - \frac{1}{2}\mathcal{H}_{\mu\dot{\nu}\dot{\rho}}\Gamma^\mu\Gamma^{\dot{\nu}\dot{\rho}}\epsilon' - \left(\frac{1}{g} + \mathcal{H}_{\dot{1}\dot{2}\dot{3}}\right)\Gamma_{\dot{1}\dot{2}\dot{3}}\epsilon' \\ &\quad - \frac{g^2}{2}\{X^{\dot{\mu}}, X^i, X^j\}\Gamma^{\dot{\mu}}\Gamma^{ij}\epsilon' + \frac{g^2}{6}\{X^i, X^j, X^k\}\Gamma^{ijk}\Gamma^{\dot{1}\dot{2}\dot{3}}\epsilon', \end{aligned} \quad (6.7)$$

$$\delta b_{\dot{\mu}\dot{\nu}} = -i(\bar{\epsilon}'\Gamma_{\dot{\mu}\dot{\nu}}\Psi'), \tag{6.8}$$

$$\delta b_{\mu\dot{\nu}} = -iV(\bar{\epsilon}'\Gamma_{\mu}\Gamma_{\dot{\nu}}\Psi') + ig(\bar{\epsilon}\Gamma_{\mu}\Gamma_{i}\Gamma_{\dot{1}\dot{2}\dot{3}}\Psi')\partial_{\dot{\nu}}X^{i}. \tag{6.9}$$

A peculiar property of this SUSY transformation is that the perturbative vacuum (the configuration with all fields vanishing) is not invariant under this transformation due to the term in $\delta\Psi'$ proportional to $1/g$. We can naturally interpret this term as a contribution of the background C-field. In the M5-brane action coupled to background fields, the self-dual field strength is defined by $H = db + C$ (up to coefficients depending on conventions). The inclusion of C-field in the field strength is required by the invariance of the action under C-field gauge transformations. The shift of the field strength $\mathcal{H}_{\dot{1}\dot{2}\dot{3}}$ by $(1/g)$ in the action as well as in the SUSY transformation suggests that the relation $C \propto g^{-1}$ between the Nambu-Poisson structure and the C-field background. This statement of course depends on the normalization of the gauge field C. For more detail about this relation, see section 7, where we derive the precise form of this relation including the numerical coefficients.

In fact, M5-brane in a constant C-field background is still 1/2 BPS. The effect of the C-field is changing which half of 32 supersymmetries remain unbroken. We can find this phenomenon in our six-dimensional theory. In addition to 16 supersymmetries we described above, the theory has 16 non-linear fermionic symmetries $\delta^{(\mathrm{nl})}$, which shift the fermion by a constant spinor

$$\delta^{(\mathrm{nl})}\Psi' = \chi, \quad \delta^{(\mathrm{nl})}X^{i} = \delta^{(\mathrm{nl})}b_{\dot{\mu}\dot{\nu}} = \delta^{(\mathrm{nl})}b_{\mu\dot{\nu}} = 0. \tag{6.10}$$

The action is invariant under this transformation because constant functions in $y^{\dot{\mu}}$ space are in the center of the 3-algebra. The perturbative vacuum is invariant under the combination of two fermionic symmetries

$$\delta_{\epsilon'} - \frac{1}{g}\delta^{(\mathrm{nl})}_{\epsilon'}. \tag{6.11}$$

In the weak coupling limit $g \to 0$, the transformation laws for this combined symmetry agree with those of an $\mathcal{N} = (2,0)$ tensor multiplet [14].

$$\delta X^{i} = i\bar{\epsilon}'\Gamma^{i}\Psi', \tag{6.12}$$

$$\delta\Psi' = \partial_{\underline{\mu}}X^{i}\Gamma^{\underline{\mu}}\Gamma^{i}\epsilon' - \frac{1}{12}H_{\underline{\mu}\underline{\nu}\underline{\rho}}\Gamma^{\underline{\mu}\underline{\nu}\underline{\rho}}\epsilon', \tag{6.13}$$

$$\delta b_{\underline{\mu}\underline{\nu}} = -i(\bar{\epsilon}'\Gamma_{\underline{\mu}\underline{\nu}}\Psi'). \tag{6.14}$$

We obtained the transformation (6.14) only for $\underline{\mu}\underline{\nu} = \dot{\mu}\dot{\nu}$ and $\mu\dot{\nu}$. To obtain the transformation law of the $b_{\mu\nu}$ components, we first compute the transformation of $\mathcal{H}_{\dot{\mu}\dot{\nu}\dot{\rho}}$ and $\mathcal{H}_{\mu\dot{\nu}\dot{\rho}}$ by using the transformation law of $b_{\dot{\mu}\dot{\nu}}$ and $b_{\mu\dot{\nu}}$. Because the field strength is self-dual, it also gives $\delta\widetilde{\mathcal{H}}_{\mu\nu\dot{\rho}}$ and $\delta\widetilde{\mathcal{H}}_{\mu\nu\rho}$. The equations of motion (5.11) and (5.12) are the Bianchi identities as well for these components of field strength. If we can solve these Bianchi identities on shell and express them by using $b_{\mu\nu}$, we can extract the transformation law of $b_{\mu\nu}$ from $\delta\widetilde{\mathcal{H}}_{\mu\nu\dot{\rho}}$ and $\delta\widetilde{\mathcal{H}}_{\mu\nu\rho}$. In the free field limit $g = 0$, we can easily carry out this procedure and obtain (6.14) for $b_{\mu\nu}$.

JHEP08(2008)014

7. Derivation of D4 action from M5

In this section we demonstrate that the double dimensional reduction of the six-dimensional theory correctly reproduces the action of non-commutative U(1) gauge theory, which is realized on a D4-brane in a B-field background.

We here recover the overall factor T_6 in the front of the action. This has mass dimension 6 and can be regarded as the tension of the five-brane, while the coupling constant g is a dimensionless parameter. We should note that this tension T_6 is not necessarily the same as the usual M5-brane tension $T_{\rm M5}$, because it may be corrected by the background C-field. We will later determine the parameters g and T_6 by comparing the five dimensional action obtained by the douple dimensional reduction of the six-dimensional theory to the non-commutative U(1) action realized on a D4-brane in a B-field background in type IIA theory. Once we obtain the expression for g and T_6 in terms of type IIA parameters, it will be easy to rewrite them in terms of the M-theory Planck scale and the magnitude of the C-field.

The double dimensional reduction means that we wrap one leg of the M5-brane on a compactified dimension, so that through Kaluza-Klein reduction we get one fewer dimension for both the target space and the world-volume. Let us choose the compactified dimension to be $X^{\dot{3}}$. In the double dimensional reduction, we suppress $y^{\dot{3}}$-dependence of all fields except $X^{\dot{3}}$. We have

$$X^{\dot{3}} = \frac{1}{g} y^{\dot{3}}, \qquad b^{\dot{3}} = 0. \tag{7.1}$$

We used a gauge symmetry generated by $\Lambda_{\dot{1}}$ and $\Lambda_{\dot{2}}$ to set $b^{\dot{3}} = 0$. We impose the periodicity condition

$$X^{\dot{3}} \sim X^{\dot{3}} + L_{11}. \tag{7.2}$$

The relation (7.1) and (7.2) implies that the compactification period of the coordinate $y^{\dot{3}}$ is gL_{11}, and thus, the overall factor of the five dimensional theory becomes $gL_{11}T_6$.

Let us now first carry out the dimensional reduction for the bosonic terms in the action. Since all the fields except $X^{\dot{3}}$ have no dependence on $y^{\dot{3}}$, we set $\partial_{\dot{3}} = 0$ unless it acts on $X^{\dot{3}}$. We will use the notation that indices $\dot{\alpha}, \dot{\beta}, \ldots$ take values in $\{\dot{1}, \dot{2}\}$, and $a, b, \ldots$ take values in $\{0, 1, 2, \dot{1}, \dot{2}\}$. The antisymmetrized tensor $\epsilon^{\dot{\alpha}\dot{\beta}}$ is defined as $\epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}\dot{3}}$.

Expecting that we will obtain a gauge field theory on a D4-brane, let us define the gauge potentials

$$\hat{a}_\mu = b_{\mu\dot{3}} \qquad \hat{a}_{\dot{\alpha}} = b_{\dot{\alpha}\dot{3}}. \tag{7.3}$$

The covariant derivatives become

$$D_\mu X^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}} \hat{F}_{\mu\dot{\beta}}, \quad D_\mu X^{\dot{3}} = -\tilde{a}_\mu, \quad D_\mu X^i = \hat{D}_\mu X^i, \tag{7.4}$$

where $\hat{F}_{ab}$, $\widetilde{a}_\mu$, and $\hat{D}_a$ are defined by

$$\hat{F}_{ab} = \partial_a \hat{a}_b - \partial_b \hat{a}_a + g\{\hat{a}_a, \hat{a}_b\}, \tag{7.5}$$

$$\tilde{a}_\mu = \epsilon^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\alpha}} b_{\mu\dot{\beta}}, \tag{7.6}$$

$$\hat{D}_\mu \Phi = \partial_\mu \Phi + g\{\hat{a}_\mu, \Phi\}. \tag{7.7}$$

The Poisson bracket $\{\cdot,\cdot\}$ is defined as the reduction of the Nambu-Poisson bracket

$$\{f,g\} = \{y^{\dot{3}}, f, g\}. \tag{7.8}$$

We note that the components $b_{\mu\dot{\beta}}$ only show up through the form $\tilde{a}_\mu$ in D4 action. Thus we find that, after double dimensional reduction, the scalar kinetic term in the BLG Lagrangian become

$$-\frac{T_6}{2}\int d^3x\langle (D_\mu X^I)^2\rangle = -\frac{gL_{11}T_6}{2}\int d^3xd^2y\left(\tilde{a}_\mu^2 + \hat{F}^2_{\mu\dot{\alpha}} + (\hat{D}_\mu X^i)^2\right). \tag{7.9}$$

The Nambu-Poisson brackets which appear in the potential terms of the BLG action are

$$\{X^{\dot{1}}, X^{\dot{2}}, X^{\dot{3}}\} = \frac{1}{g^2}\hat{F}_{\dot{1}\dot{2}} + \frac{1}{g^3}, \tag{7.10}$$

$$\{X^{\dot{3}}, X^{\dot{\alpha}}, X^i\} = \frac{1}{g^2}\epsilon^{\dot{\alpha}\dot{\beta}}\hat{D}_{\dot{\beta}}X^i, \tag{7.11}$$

$$\{X^{\dot{3}}, X^i, X^j\} = \frac{1}{g}\{X^i, X^j\}. \tag{7.12}$$

The potential term becomes

$$\begin{aligned}&-\frac{T_6}{12}\int d^3x\langle g^4\{X^I,X^J,X^K\}^2\rangle \\ &\quad = gL_{11}T_6\int d^3xd^2y\left[-\frac{1}{2}\left(\hat{F}_{\dot{1}\dot{2}}+\frac{1}{g}\right)^2 - \frac{1}{2}(D_{\dot{\alpha}}X^i)^2 - \frac{g^2}{4}\{X^i,X^j\}^2\right].\end{aligned} \tag{7.13}$$

Upon integration over the base space and removing total derivatives, we can replace $(\hat{F}_{\dot{1}\dot{2}}+1/g)^2$ by $\hat{F}^2_{\dot{1}\dot{2}}+1/g^2$.

It is also straightforward to show that the Chern-Simons term (5.3) gets simplified considerably as

$$-\frac{gL_{11}T_6}{2}\int d^3xd^2y\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}\tilde{a}_\lambda. \tag{7.14}$$

Here again the action depends on $b_{\mu\dot{\beta}}$ only through $\tilde{a}_\mu$. As the action depends on the field $\tilde{a}_\mu$ only algebraically (namely without derivative), we can integrate it out. There are only two terms involving $\tilde{a}_\mu$ and by completing square, we find that the effect of integrating out $\tilde{a}_\mu$ is to replace all terms involving $\tilde{a}_\mu$ by

$$-\frac{gL_{11}T_6}{4}\int d^3xd^2y\hat{F}^2_{\mu\nu}. \tag{7.15}$$

The fermion part can be evaluated similarly. The covariant derivatives and bracket are

$$\Gamma^\mu D_\mu\Psi' = \Gamma^\mu(\partial_\mu\Psi' + g\left\{a_\mu,\Psi'\right\}) := \Gamma^\mu\hat{D}_\mu\Psi', \tag{7.16}$$

$$\frac{1}{2}\Gamma_{IJ}\{X^I,X^J,\Psi'\} = \Gamma_{\dot{\alpha}}\Gamma_{\dot{1}\dot{2}\dot{3}}\hat{D}_{\dot{\beta}}\Psi' + \Gamma_{\dot{3}}\Gamma_i\left\{X^i,\Psi'\right\}, \tag{7.17}$$

$$\hat{D}_{\dot{\beta}}\Psi' := \partial_{\dot{\beta}}\Psi' + g\{a_{\dot{\beta}},\Psi'\}. \tag{7.18}$$

It is quite remarkable that, after collecting all the kinetic, potential and Chern-Simons temrs, the 4+1 dimensional Lorentz invariance is restored (up to the breaking by the non-commutativity). The sum of all these terms is simply

$$gL_{11}T_6 \int d^3x d^2y \left[-\frac{1}{2}(\hat{D}_a X^i)^2 - \frac{1}{4}\hat{F}_{ab}^2 - \frac{g^2}{4}\{X^i, X^j\}^2 - \frac{1}{2g^2} \right.$$
$$\left. + \frac{i}{2}\left(\bar{\Psi}''\Gamma^a \hat{D}_a \Psi'' + g\bar{\Psi}''\Gamma_i\{X^i, \Psi''\} \right) \right]. \tag{7.19}$$

We performed the unitary transformation $\Psi' = (1/\sqrt{2})(\Gamma_{\dot{3}} + \Gamma^7)\Psi''$ to obtain the correct chirality condition $\Gamma_{\dot{3}}\Psi'' = -\Psi''$ for the gaugino on the D4-brane. (Nore that $\dot{3}$ is now the "eleventh" direction and $\Gamma_{\dot{3}}$ is the chirality matrix in IIA theory.)

Let us compare the action (7.19) with the known result [15, 16] for a D4-brane in a B-field background, and match the parameters in this theory and those in type IIA string theory. The non-commutative gauge theory on D4-brane in a B-field background is described with the Moyal product $*$, and the corresponding commutator, the so-called Moyal bracket $[\cdot, \cdot]_{\text{Moyal}}$, defined by

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial \xi^i}\frac{\partial}{\partial \zeta^j}} f(x+\xi)g(x+\zeta)|_{\xi=\zeta=0}, \tag{7.20}$$

$$[f, g]_{\text{Moyal}} = f * g - g * f = \theta^{ij}\partial_i f \partial_j g + \mathcal{O}(\theta^3). \tag{7.21}$$

The non-commutativity parameter θ^{ij} has the dimension of (length)^2. Because the action (7.19) includes only finite powers of derivatives, it should be compared to the weak coupling limit $\theta \to 0$ of the non-commutative gauge theory. These two match if we truncate the Moyal bracket into the Poisson bracket by

$$[f, g]_{\text{Moyal}} \to \frac{\theta}{T_{\text{str}}}\{f, g\}, \tag{7.22}$$

where we turn on the non-commutativity in the $\dot{1}$-$\dot{2}$ directions by setting

$$\theta^{\dot{1}\dot{2}} = \frac{\theta}{T_{\text{str}}}, \quad \theta^{\mu\dot{\alpha}} = \theta^{\mu\nu} = 0. \tag{7.23}$$

Note that θ is defined as a dimensionless parameter. In the small θ limit, the bosonic part of the action of the non-commutative U(1) gauge theory on a D4-brane is given by [15, 16]

$$S = \frac{T_{D4}}{\theta} \int d^3x d^2y \left[-\frac{1}{2}(D_a X^i)^2 - \frac{1}{4T_{\text{str}}}F_{ab}^2 - \frac{\theta^2}{4}\{X^i, X^j\}^2 - \frac{1}{2\theta^2} \right], \tag{7.24}$$

in the open string frame. The world-volume coordinate $y^{\dot{\alpha}}$ in the open string frame is related to the target space coordinates $X^{\dot{\alpha}}$ by

$$X^{\dot{\alpha}} = \frac{1}{\theta}y^{\dot{\alpha}}. \tag{7.25}$$

The covariant derivative and the field strength are

$$D_a X^i = \partial_\mu X^i + \frac{\theta}{T_{\text{str}}}\{A_a, X^i\}, \quad F_{ab} = \partial_a A_b - \partial_b A_a + \frac{\theta}{T_{\text{str}}}\{A_a, A_b\}. \tag{7.26}$$

We normalize the gauge field A_a so that it couples to the string endpoints by charge 1 through the boundary coupling $S = \int_{\partial \mathrm{F1}} A$ of the fundamental string world-sheet, and this gauge field has mass dimension 1. In the weak coupling limit, the non-commutativity parameter θ is related to the background B-field by

$$B = T_{\mathrm{str}} \theta dX^{\dot{1}} \wedge dX^{\dot{2}} = \frac{T_{\mathrm{str}}}{\theta} dy^{\dot{1}} \wedge dy^{\dot{2}}. \tag{7.27}$$

By comparing two actions (7.19) and (7.24), we obtain the following relations among parameters:

$$T_6 = \frac{T_{M5}}{\theta^2}, \tag{7.28}$$

$$g = \theta. \tag{7.29}$$

To relate quantities in IIA and M-theory, we use the following relations among tensions of M-branes and IIA-branes.

$$T_{D4} = L_{11} T_{M5}, \quad T_{\mathrm{str}} = L_{11} T_{D2} = L_{11} T_{M2}. \tag{7.30}$$

The relation $T_{M2}^2 = 2\pi T_{M5}$ is also useful.

In addition to the agreement of the action through the relations (7.28) and (7.29), we can check the consistency in some places.

Firstly, the relation (7.25) between the world-volume coordinates and the target space coordinates can naturally be lifted to the relation (4.7).

Secondly, the overall factor T_6 agrees with the effective tension of M2-branes induced by the background C-field. The background B-field (7.27) is lifted to the background three-form field

$$C_3 = \theta T_{\mathrm{M2}} dX^{\dot{1}} \wedge dX^{\dot{2}} \wedge dX^{\dot{3}} = \frac{T_{\mathrm{M2}}}{\theta^2} dy^{\dot{1}} \wedge dy^{\dot{2}} \wedge dy^{\dot{3}}. \tag{7.31}$$

(We use the convention in which the gauge fields B and C couple to the world-volume of corresponding branes by charge 1 through the couplings $\int_{\mathrm{F1}} B$ and $\int_{\mathrm{M2}} C$.) Each flux quantum of this background field induces the charge of a single M2-brane on the M5-brane, and effective M2-brane density in the y-space is $\theta^{-2} T_{\mathrm{M2}}/(2\pi)$. Thus, if we assume that the tension of M5-brane is dominated by the induced M2-branes, the effective tension becomes $T_{\mathrm{M2}} \times \theta^{-2} T_{\mathrm{M2}}/(2\pi) = \theta^{-2} T_{\mathrm{M5}}$. This agrees with the overall coefficients T_6 given in the relation (7.28).

Finally, the charge of the self-dual strings is consistent with the Dirac's quantization condition. From the comparison of the actions we obtain the relation of gauge fields

$$\hat{a}_a = \frac{1}{T_{\mathrm{str}}} A_a. \tag{7.32}$$

As we mentioned above, the gauge field A couples to string endpoints by charge 1. By the correspondence (7.32) we can determine the strength of the coupling of $\hat{a}$ and b to boundaries of the corresponding branes. The boundary interactions are given by

$$S = T_{\mathrm{str}} \int_{\partial \mathrm{F1}} \hat{a} = \frac{T_{\mathrm{M2}}}{\theta} \int_{\partial \mathrm{M2}} b. \tag{7.33}$$

To obtain the second equality in (7.33), we used the fact that a string endpoint is lifted to an M2-brane boundary wrapped on the S^1 along $y^{\dot{3}}$ with period gL_{11}. The coupling (7.33) shows that the charge of self-dual strings (boundary of M2-branes ending on the M5-brane) is $Q = \theta^{-1}T_{\mathrm{M2}}$. Because the gauge field b is a self-dual field, Q is the electric charge as well as the magnetic charge of a self-dual string, and it must satisfy the Dirac's quantization condition

$$\frac{Q^2}{T_6} = 2\pi. \tag{7.34}$$

We can easily check that this relation certainly holds.

We can now explain the constant shift in the field strength as follows. The M2-brane action includes the following coupling to the bulk 3-form field C and the self-dual 2-form field b:

$$S_{\mathrm{M2}} = \int_{\mathrm{M2}} C_3 + \frac{T_{\mathrm{M2}}}{\theta}\int_{\partial\mathrm{M2}} b. \tag{7.35}$$

The gauge invariance of this action requires that under the gauge transformation $\delta C_3 = d\alpha_2$, the self-dual field on the M5-brane must transform as $\delta b_2 = -\alpha_2/(\theta^{-1}T_{\mathrm{M2}})$. Thus, the gauge invariant field strength H of the tensor field b should be defined by

$$H = db + \frac{\theta}{T_{M2}}C. \tag{7.36}$$

Therefore, the background gauge field (7.31) shifts the field strength as

$$H = db + \frac{1}{\theta}dy^{\dot{1}} \wedge dy^{\dot{2}} \wedge dy^{\dot{3}}. \tag{7.37}$$

This is the same as the constant shift in the definition (4.24) of $\mathcal{H}_{\dot{1}\dot{2}\dot{3}}$.

Now we have relations between parameters in the BLG theory and those in M-theory. The D4-brane action obtained by the double dimensional reduction is the weak coupling ($g = \theta \to 0$) limit of non-commutative U(1) theory because the Moyal bracket is replaced by the Poisson bracket. The coupling constant is determined by the background C-field, and the weak coupling means strong C-field background through the relation (7.31). Our M5-brane theory is expected to apply better to the limit of large C-field background. This is also confirmed in the comparison of the five-brane tension. As we mentioned above, the effective tension T_6 is dominated by the tension of M2-branes induced by the background C-field. This is the case when the background C-field is very large.

For a finite C-field background, we expect that the Nambu-Poisson bracket should be replaced by a quantum Nambu bracket.

8. Seiberg-Witten map

It was found by Seiberg and Witten [16] that the gauge symmetry on a noncommmutative space can be matched with the gauge symmetry on a classical space via the so-called Seiberg-Witten map

$$\hat{\delta}_{\hat{\lambda}}\hat{\Phi}(\Phi) = \hat{\Phi}(\Phi + \delta_\lambda\Phi) - \hat{\Phi}(\Phi), \tag{8.1}$$

JHEP08(2008)014

where $\hat{\Phi}(\Phi)$ is the field variable, and $\hat{\delta}_{\hat{\lambda}}$ the gauge transformation in the noncommutative gauge theory corresponding to Φ and δ_λ living on the classical space. The gauge transformation parameter $\hat{\lambda}(A, \lambda)$ in the noncommutative gauge theory is a function of the gauge potential A and gauge transformation parameter λ in the gauge theory on classical spacetime. The Seiberg-Witten map is found as an infinite expansion of the noncommutativity parameters.

In this section we find the Seiberg-Witten map connecting the gauge theories on spacetimes with and without the Nambu-Poisson structure, corresponding to M5-brane theories in trivial or constant C-field background. In this section only, we denote all variables in our M5-brane theory by symbols with hats, and those in trivial backgrounds by symbols without hats. As $g \to 0$, the variables with hats should reduce to those without hats.

In the trivial background, we have the gauge fields $b_{\dot{\mu}\dot{\nu}}$, $b_{\mu\dot{\mu}}$ and gauge transformations

$$\delta_\Lambda b_{\dot{\mu}\dot{\nu}} = \partial_{\dot{\mu}}\Lambda_{\dot{\nu}} - \partial_{\dot{\nu}}\Lambda_{\dot{\mu}}, \qquad \delta_\Lambda b_{\mu\dot{\mu}} = \partial_\mu \Lambda_{\dot{\mu}} - \partial_{\dot{\mu}}\Lambda_\mu. \tag{8.2}$$

In the M5-brane theory with a C-field background, we have

$$\hat{\delta}_{\hat{\Lambda}} \hat{b}_{\dot{\mu}\dot{\nu}} = \partial_{\dot{\mu}}\hat{\Lambda}_{\dot{\nu}} - \partial_{\dot{\nu}}\hat{\Lambda}_{\dot{\mu}} + g\hat{\kappa}^{\dot{\lambda}}\partial_{\dot{\lambda}}\hat{b}_{\dot{\mu}\dot{\nu}}, \tag{8.3}$$

$$\hat{\delta}_{\hat{\Lambda}} \hat{b}_{\mu\dot{\mu}} = \partial_\mu \hat{\Lambda}_{\dot{\mu}} - \partial_{\dot{\mu}}\hat{\Lambda}_\mu + g\hat{\kappa}^{\dot{\nu}}\partial_{\dot{\nu}}\hat{b}_{\mu\dot{\mu}} + g(\partial_{\dot{\mu}}\hat{\kappa}^{\dot{\nu}})\hat{b}_{\mu\dot{\nu}}. \tag{8.4}$$

It will be convenient to use the following variables

$$\hat{b}^{\dot{\mu}} \equiv \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}}\hat{b}_{\dot{\nu}\dot{\lambda}}, \qquad \hat{B}_\mu{}^{\dot{\mu}} \equiv \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}}\partial_{\dot{\nu}}\hat{b}_{\mu\dot{\lambda}}. \tag{8.5}$$

instead of $\hat{b}_{\dot{\mu}\dot{\nu}}$ and $\hat{b}_{\mu\dot{\mu}}$. Similarly we define $b^{\dot{\mu}}$ and $B_\mu{}^{\dot{\mu}}$ in the same way. We shall impose the constraints

$$\partial_{\dot{\mu}}\hat{B}_\mu{}^{\dot{\mu}} = 0, \qquad \text{and} \qquad \partial_{\dot{\mu}}B_\mu{}^{\dot{\mu}} = 0 \tag{8.6}$$

on $\hat{B}_\mu{}^{\dot{\mu}}$ and $B_\mu{}^{\dot{\mu}}$ so that the existence of $\hat{b}_{\mu\dot{\mu}}$ and $b_{\mu\dot{\mu}}$ is guaranteed when the former are given. The gauge transformations of the new variables are

$$\hat{\delta}_{\hat{\Lambda}}\hat{b}^{\dot{\mu}} = \hat{\kappa}^{\dot{\mu}} + g\hat{\kappa}^{\dot{\nu}}\partial_{\dot{\nu}}\hat{b}^{\dot{\mu}}, \qquad \hat{\delta}_{\hat{\Lambda}}\hat{B}_\mu{}^{\dot{\mu}} = \partial_\mu\hat{\kappa}^{\dot{\mu}} + g\hat{\kappa}^{\dot{\nu}}\partial_{\dot{\nu}}\hat{B}_\mu{}^{\dot{\mu}} - g(\partial_{\dot{\nu}}\hat{\kappa}^{\dot{\mu}})\hat{B}_\mu{}^{\dot{\nu}}, \tag{8.7}$$

where

$$\hat{\kappa}^{\dot{\mu}} \equiv \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}}\partial_{\dot{\nu}}\hat{\Lambda}_{\dot{\lambda}} \tag{8.8}$$

was denoted by $\delta_\Lambda y^{\dot{\mu}}$ above, and it satisfies

$$\partial_{\dot{\mu}}\hat{\kappa}^{\dot{\mu}} = 0. \tag{8.9}$$

Analogous to all these equations above, we have the corresponding equations for variables on a M5-brane in the trivial background. They can be obtained by taking the $g \to 0$ limit as

$$\delta_\Lambda b^{\dot{\mu}} = \kappa^{\dot{\mu}}, \qquad \delta_\Lambda B_\mu{}^{\dot{\mu}} = \partial_\mu\kappa^{\dot{\mu}}, \qquad \kappa^{\dot{\mu}} = \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}}\partial_{\dot{\nu}}\Lambda_{\dot{\lambda}}. \tag{8.10}$$

We need two Seiberg-Witten maps $\hat{b}^{\dot{\mu}}(b)$ and $\hat{B}_\mu{}^{\dot{\mu}}(B, b)$ satisfying

$$\hat{\delta}_{\hat{\Lambda}}\hat{b}^{\dot{\mu}}(b) = \hat{b}^{\dot{\mu}}(b + \delta_\Lambda b) - \hat{b}^{\dot{\mu}}(b), \tag{8.11}$$

$$\hat{\delta}_{\hat{\Lambda}}\hat{B}_\mu{}^{\dot{\mu}}(B, b) = \hat{B}_\mu{}^{\dot{\mu}}(B + \delta_\Lambda B, b + \delta_\Lambda b) - \hat{B}_\mu{}^{\dot{\mu}}(B, b). \tag{8.12}$$

The solutions would be infinite expansions in g. To the first order terms in g, we find the solution as

$$\hat{b}^{\dot{\mu}}(b) = b^{\dot{\mu}} + \frac{g}{2} b^{\dot{\nu}} \partial_{\dot{\nu}} b^{\dot{\mu}} + \frac{g}{2} b^{\dot{\mu}} \partial_{\dot{\nu}} b^{\dot{\nu}} + \mathcal{O}(g^2), \tag{8.13}$$

$$\begin{aligned}\hat{B}_{\mu}{}^{\dot{\mu}}(B, b) = & B_{\mu}{}^{\dot{\mu}} + g b^{\dot{\nu}} \partial_{\dot{\nu}} B_{\mu}{}^{\dot{\mu}} - \frac{g}{2} b^{\dot{\nu}} \partial_{\mu} \partial_{\dot{\nu}} b^{\dot{\mu}} + \frac{g}{2} b^{\dot{\mu}} \partial_{\mu} \partial_{\dot{\nu}} b^{\dot{\nu}} + g \partial_{\dot{\nu}} b^{\dot{\nu}} B_{\mu}{}^{\dot{\mu}} \\ & - g \partial_{\dot{\nu}} b^{\dot{\mu}} B_{\mu}{}^{\dot{\nu}} - \frac{g}{2} \partial_{\dot{\nu}} b^{\dot{\nu}} \partial_{\mu} b^{\dot{\mu}} + \frac{g}{2} \partial_{\dot{\nu}} b^{\dot{\mu}} \partial_{\mu} b^{\dot{\nu}} + \mathcal{O}(g^2),\end{aligned} \tag{8.14}$$

$$\hat{\kappa}^{\dot{\mu}}(\kappa, b) = \kappa^{\dot{\mu}} + \frac{g}{2} b^{\dot{\nu}} \partial_{\dot{\nu}} \kappa^{\dot{\mu}} + \frac{g}{2} (\partial_{\dot{\nu}} b^{\dot{\nu}}) \kappa^{\dot{\mu}} - \frac{g}{2} (\partial_{\dot{\nu}} b^{\dot{\mu}}) \kappa^{\dot{\nu}} + \mathcal{O}(g^2). \tag{8.15}$$

Some of the coefficients here are not completely fixed by the Seiberg-Witten map conditions (8.11)–(8.12), but they are uniquely determined by the requirement that the constraint (8.6) is preserved by the Seiberg-Witten map. It should be possible to solve for higher order terms order by order.

To be complete, let us consider X^I and Ψ, or anything that transforms like

$$\delta \hat{\Phi} = g \hat{\kappa}^{\dot{\mu}} \partial_{\dot{\mu}} \hat{\Phi}. \tag{8.16}$$

The classical counterpart of $\hat{\Phi}$ has

$$\delta \Phi = 0. \tag{8.17}$$

The Seiberg-Witten map condition (8.1) is easy to solve for this case to obtain the first order terms, and the solution is

$$\hat{\Phi} = \Phi + g b^{\dot{\mu}} \partial_{\dot{\mu}} \Phi + \mathcal{O}(g^2). \tag{8.18}$$

A comment is needed here regarding the map between $\hat{\kappa}^{\dot{\mu}}$ and $\kappa^{\dot{\mu}}$, which are defined in terms of the gauge transformation parameters $\hat{\Lambda}_{\dot{\mu}}$ and $\Lambda_{\dot{\mu}}$. If one wants to determine the map between $\hat{\Lambda}_{\dot{\mu}}$ and $\Lambda_{\dot{\mu}}$, it is necessary to fix the ambiguity in the gauge parameters for a given gauge transformation. It is obvious that the transformation of the gauge parameters

$$\hat{\Lambda}_{\dot{\mu}} \to \hat{\Lambda}_{\dot{\mu}} + \partial_{\dot{\mu}} \hat{\xi}, \qquad \hat{\Lambda}_{\mu} \to \hat{\Lambda}_{\mu} + \partial_{\mu} \hat{\xi}, \tag{8.19}$$

$$\Lambda_{\dot{\mu}} \to \Lambda_{\dot{\mu}} + \partial_{\dot{\mu}} \xi, \qquad \Lambda_{\mu} \to \Lambda_{\mu} + \partial_{\mu} \xi \tag{8.20}$$

does not change the gauge transformations (8.2)–(8.4) at all. To avoid this ambiguity in the gauge transformation parameters, we can use $\hat{\kappa}^{\dot{\mu}}$ and $\kappa^{\dot{\mu}}$ instead, and the existence of $\hat{\Lambda}_{\dot{\mu}}$ and $\Lambda_{\dot{\mu}}$ are guaranteed by the constraints (8.9). One can check that the constraint (8.9) is preserved by the Seiberg-Witten map (8.15).

The ambiguity involved here is in the same form as a gauge transformation of 1-form gauge fields, and hence we can "gauge fix" the gauge transformation parameters by the following constraints

$$\partial_{\dot{\mu}} \hat{\Lambda}^{\dot{\mu}} = 0, \qquad \partial_{\dot{\mu}} \Lambda^{\dot{\mu}} = 0. \tag{8.21}$$

(Here we are raising and lowering indices using the metric $\delta_{\dot{\mu}\dot{\nu}}$ on $\mathcal{N}$.) We can solve these constraints by

$$\hat{\Lambda}^{\dot{\mu}} = \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\nu}} \hat{\eta}_{\dot{\lambda}}, \qquad \Lambda^{\dot{\mu}} = \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\nu}} \eta_{\dot{\lambda}}, \tag{8.22}$$

JHEP08(2008)014

and again we can demand the constraints

$$\partial_{\dot\mu}\hat\eta^{\dot\mu} = 0, \qquad \partial_{\dot\mu}\eta^{\dot\mu} = 0 \tag{8.23}$$

on the parameters $\hat\eta^{\dot\mu}$ and $\eta^{\dot\mu}$, in terms of which we have

$$\hat\kappa^{\dot\mu} = -\partial^2\hat\eta^{\dot\mu}, \qquad \kappa^{\dot\mu} = -\partial^2\eta^{\dot\mu}, \tag{8.24}$$

where $\partial^2 \equiv \partial_{\dot\mu}\partial^{\dot\mu}$ is the Laplace operator on $\mathcal{N}$. This allows us to deduce the Seiberg-Witten map between $\hat\eta^{\dot\mu}$ and $\eta^{\dot\mu}$, and finally the Seiberg-Witten map for the gauge transformation parameters can be expressed in the following nonlocal form

$$\hat\Lambda^{\dot\mu} = \Lambda^{\dot\mu} - \frac{g}{2}\epsilon^{\dot\mu\dot\nu\dot\lambda}\partial^{-2}\partial_{\dot\nu}[b^{\dot\rho}\partial_{\dot\rho}\kappa_\lambda + (\partial_{\dot\rho}b^{\dot\rho})\kappa_\lambda - (\partial_{\dot\rho}b_{\dot\lambda})\kappa^{\dot\rho}] + \cdots . \tag{8.25}$$

9. Interpretations of the M5-brane theory

9.1 As a field theory of the Nambu-Poisson structure

The gauge symmetry of the M5 world-volume theory is the volume-preserving diffeomorphism on $\mathcal{N}$. The transformation law for both X^i and Ψ are given in the same form

$$\delta_\Lambda\Phi = g\delta_\Lambda y^{\dot\mu}\partial_{\dot\mu}\Phi, \tag{9.1}$$

where the volume-preserving coordinate transformation

$$\delta_\Lambda y^{\dot\mu} = \epsilon^{\dot\mu\dot\nu\dot\lambda}\partial_{\dot\nu}\Lambda_{\dot\lambda} \tag{9.2}$$

is parametrized by three arbitrary functions $\Lambda_{\dot\mu}$.

In the above we have considered $b_{\mu\dot\mu}$ and $b_{\dot\mu\dot\nu}$ as the gauge fields for the volume-preserving diffeomorphisms. Here we give a geometrical interpretation to these quantities in terms of deformations of the Nambu-Poisson structure.

The degrees of freedom corresponding to $b_{\dot\mu\dot\nu}$, or equivalently $b^{\dot\mu}$, is easy to understand. It arises in $X^{\dot\mu}$ (4.7), and $y^{\dot\mu}$ is a longitudinal coordinate on the M_5 brane. Thus $b^{\dot\mu}$ corresponds to a coordinate transformation on $\mathcal{N}$, which is not necessarily volume-preserving because $\partial_{\dot\mu}b^{\dot\mu}$ may be nonzero. In fact, one can view $b^{\dot\mu}$ as a parametrization of the deformations of the Nambu-Poisson structure due to a change of the coordinates

$$y^{\dot\mu} \quad \rightarrow \quad gX^{\dot\mu} = y^{\dot\mu} + gb^{\dot\mu}. \tag{9.3}$$

That is,

$$\{f,g,h\} = \epsilon^{\dot\mu\dot\nu\dot\lambda}\partial_{\dot\mu}f\partial_{\dot\nu}g\partial_{\dot\lambda}h \quad \rightarrow \quad g^{-3}\epsilon^{\dot\mu\dot\nu\dot\lambda}\frac{\partial}{\partial X^{\dot\mu}}f\frac{\partial}{\partial X^{\dot\nu}}g\frac{\partial}{\partial X^{\dot\lambda}}h. \tag{9.4}$$

While $b^{\dot\mu}$ is used to parametrize deformations of the Nambu-Poisson structure, infinitesimal coordinate transformations

$$\delta b^{\dot\mu} = \delta y^{\dot\mu} + g\delta y^{\dot\nu}\partial_{\dot\nu}b^{\dot\mu} \overset{g\to 0}{\longrightarrow} \delta y^{\dot\mu}, \tag{9.5}$$

which preserve the Nambu-Poisson bracket, should be regarded as gauge transformations

The other gauge potential $b_{\mu\dot{\mu}}$ appears in the covariant derivative (4.20)

$$D_\mu = \partial_\mu - g B_\mu{}^{\dot{\mu}} \partial_{\dot{\mu}}, \tag{9.6}$$

where

$$B_\mu{}^{\dot{\mu}} = \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\nu}} b_{\mu\dot{\lambda}}. \tag{9.7}$$

Formally, the expression of the D_μ suggests that $B_\mu{}^{\dot{\mu}}$ is the gauge potential and $\partial_{\dot{\mu}}$ is the gauge symmetry generator, which also appears in (9.1). Indeed, instead of defining $B_\mu{}^{\dot{\mu}}$ in terms of $b_{\mu\dot{\mu}}$, one can view $B_\mu{}^{\dot{\mu}}$ as the fundamental gauge potential, and guarantee the existence of $b_{\mu\dot{\mu}}$ through the constraint

$$\partial_{\dot{\mu}} B_\mu{}^{\dot{\mu}} = 0. \tag{9.8}$$

The gauge field $B_\mu{}^{\dot{\mu}}$ is reminiscent of the gauge field parametrizing complex structure deformations on a Calabi-Yau 3-manifold in the Kodaira-Spencer theory [9].

Consider a generic 6 dimensional space equipped with a Nambu-Poisson structure. The decomposability of the Nambu-Poisson bracket implies that locally we can always choose 3 coordinates $y^{\dot{\mu}}$ such that the Nambu-Poisson bracket is just the Jacobian factor (3.3). Thus the Nambu-Poisson structure induces the separation of local coordinates into the two sets $\{x^\mu\}$ and $\{y^{\dot{\mu}}\}$. This is analogous to the situation of a complex manifold, for which there are holomorphic z^i and anti-holomorphic $\bar{z}^{\bar{i}}$ coordinates. A deformation of the complex structure can be described by specifying how the notion of holomorphicity is changed. A function on the complex manifold is holomorphic if

$$\bar{\partial} f = 0 \to \partial_{\bar{i}} f = 0. \tag{9.9}$$

When the complex structure is deformed, the anti-holomorphic exterior derivative is changed

$$\bar{\partial} \to \bar{\partial}_A = d\bar{z}^{\bar{i}} D_{\bar{i}}, \tag{9.10}$$

where

$$D_{\bar{i}} = \partial_{\bar{i}} + A_{\bar{i}}{}^i \partial_i. \tag{9.11}$$

The gauge potential $A_{\bar{i}}{}^i$ parametrizes how much mixing occurs between the holomorphic and anti-holomorphic coordinates due to the deformation of complex structure. In our M5-brane theory, $B_\mu{}^{\dot{\mu}}$ plays a similar role as $A_{\bar{i}}{}^i$, and the covariant derivative D_μ can be viewed as a deformation of the derivative with respect to x^μ. It should be understood as a specification of how much the coordinates x^μ and $y^{\dot{\mu}}$ are mixed by a deformation of the Nambu-Poisson structure. While the Kodaira-Spencer theory [9] is a dynamical theory of the complex structure, the M5-brane theory can be understood as a dynamical theory of the Nambu-Poisson structure.

To be more persuasive, we can make the analogy between the Kodaira-Spencer theory of complex structure and the M5-brane theory of Nambu-Poisson structure more explicit. For a Calabi-Yau 3-fold, there is a unique holomorphic (3, 0)-form $\Omega = \frac{1}{3!}\Omega_{ijk} dz^i dz^j dz^k$. This allows us to impose a constraint on the gauge potential A as

$$\partial A' = 0 \Leftrightarrow \Omega^{ijk} \partial_i A_{\bar{i}jk} = 0, \tag{9.12}$$

where

$$A' = (\Omega \cdot A), \qquad \text{i.e.} \qquad A'_{\bar{i}jk} = \Omega_{ijk} A_{\bar{i}}{}^{i}. \tag{9.13}$$

The $(3,0)$-form Ω is analogous to the Nambu-Poisson tensor field, which is taken to be $g\epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}}$ by suitably choosing the coordinates $y^{\dot{\mu}}$. If we carry out the substitution

$$A \to B, \qquad \Omega_{ijk} \to g\epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}}, \qquad z^i \to y^{\dot{\mu}}, \tag{9.14}$$

the constraint (9.12) is precisely the constraint (9.8) which guarantees the existence of $b_{\mu\dot{\mu}}$.

Furthermore, the Kodaira-Spencer equation [17], which is equivalent to the nilpotency condition of the deformed anti-holomorphic exterior derivative (9.10), is

$$\bar{\partial}_A \bar{\partial}_A = 0, \quad \Leftrightarrow \quad [D_{\bar{i}}, D_{\bar{j}}] = 0. \tag{9.15}$$

If we turn off all other fields X^i, Ψ and $b^{\dot{\mu}}$, the equation of motion for $B_\mu{}^{\dot{\mu}}$ is

$$[D_\mu, D_\nu] = 0. \tag{9.16}$$

This is exactly what one obtains from (9.15) via the replacement (9.14).

To summarize, while $b_{\mu\dot{\mu}}$ and $b_{\dot{\mu}\dot{\nu}}$ are viewed as the gauge potentials for the gauge symmetry of coordinate transformations preserving a given Nambu-Poisson structure, $B_\mu{}^{\dot{\mu}}$ and $b^{\dot{\mu}}$ should be viewed as two types of deformation parameters of the Nambu-Poisson structure of the M5-brane world-volume. We have $b^{\dot{\mu}}$ specifying the change of the Nambu-Poisson structure due to a change of coordinates $\delta y^{\dot{\mu}}$ in $\mathcal{N}$ (so that the volume form is changed), and $B_\mu{}^{\dot{\mu}}$ the change due to a mixing of the two classes of coordinates x^μ and $y^{\dot{\mu}}$. The gauge symmetry corresponds to redundant descriptions of deformations of the Nambu-Poisson structure. The M5-brane theory with a self-dual gauge field can thus be interpreted as a dynamical theory of the Nambu-Poisson structure.

9.2 As an effective theory in large C-field background

The M5-brane action obtained from the BL action with the Nambu-Poisson algebra should be interpreted as the M5-brane theory in a large C-field background. In section 6, we find this interpretation to be consistent with the properties of the supersymmetry. Furthermore, in section 7, the noncommutative D4-brane action obtained via a double dimensional reduction from the M5 theory has $g \sim \theta$. As it is well known that for a D4-brane the noncommutativity parameter θ is given by B^{-1} in the large B-field background, we deduce that (with the specific normalization of C-field such that the self-dual field strength becomes $H = db + C$)

$$g \sim C^{-1} \tag{9.17}$$

in the large C-field background for our M5-brane theory.

In [1], an analogy was made between the Nambu-Poisson structure on M5-brane and the Poisson structure on D-branes. For a D-brane in a constant B-field background, the effective D-brane theory is best described as a noncommutative field theory. In the limit of both large and small B-field background, the noncommutativity is small and the commutator can be approximated by a Poisson bracket, and the Poisson structure is determined by the two-form B-field background. More precisely, the gauge-invariant quantity which should be used to specify the background is $\mathcal{F} = B + F$, where $F = dA$ is the field strength of a gauge field A in the D-brane world-volume theory. We can fix the gauge so that the background value of $\mathcal{F} = B$. Then a nontrivial configuration of the gauge field A corresponds to a change of $\mathcal{F}$, and thus a change of the Poisson structure.

The invariant self-dual 3-form field strength on the M5-brane is $H = C + db$, where b is the 2-form gauge potential on the M5-brane. The Nambu-Poisson structure determined by a given background $H = C$ is therefore deformed by turning on b, while a gauge transformations of b preserves H and thus the Nambu-Poisson structure.

In the case of D-branes in B-field background, there are several different ways to verify the connection between B-field background and the noncommutativity. One way is to quantize an open string ending on a D-brane and check that the endpoint coordinates obey a commutation relation determined by the B-field background [18]. Or one can compute open string scattering amplitudes [19]. Another way [16] is to find the Seiberg-Witten map which maps the commutative field A to the noncommutative field $\hat{A}$, and then check that the (commutative) D-brane field theory with B-field background explicitly turned on is approximately the same as the noncommutative field theory without explicit B-field background.

Quantization of open membranes in the large C-field background has been extensively studied in the literature [20, 21]. However, quantization is by its nature associated with a Poisson structure, and thus the appearance of a Nambu-Poisson structure can not be manifest. On the other hand, in [22], the scattering amplitudes of open membranes were studied in a large C-field background, and the result indicated that indeed the C-field background induces a Nambu-Poisson structure.

As a further support of our interpretation of the Nambu-Poisson structure as an effect of the C-field background, in the previous section we found the Seiberg-Witten map which matches the gauge transformation of ordinary M5-brane theory with the deformed gauge transformation (4.14)–(4.17) supposedly corresponding to a C-field background.

10. Further remarks

Quantization of the coefficients of Chern-Simons term. In section 3 we showed that if the internal space is $\mathcal{N} = \mathbf{R}^3$ the model has no coupling constant at all. What happens when $\mathcal{N}$ is a non-trivial space? In such a case there may not be any simple way to remove the coupling constant by re-scaling of fields. It is an interesting problem to clarify constraints imposed on this coupling constant. For the case when the 3-algebra is taken to be $\mathcal{A}_4$, Bagger and Lambert [2] have shown that the eigenvalues of the structure constant $f^{abcd} \sim \epsilon^{abcd}$ must be quantized as $\lambda = \pi/k$ for $k = 1, 2, 3 \cdots$. It implies that in BLG model there are no tunable continuous parameters in the theory. In this paper, we have used Lie 3-algebras which has an infinite number of generators. One may wonder if we might have a similar constraint for the structure constant, especially if the internal space $\mathcal{N}$ is a compact space. For $\mathcal{N} = T^3$, for example, the generators are labeled by $\vec{n} \in \mathbf{Z}^3$ and the structure constant becomes [1]

$$f^{\vec{n}_1\vec{n}_2\vec{n}_3\vec{n}_4} = \frac{\alpha}{V}\,\vec{n}_1\cdot(\vec{n}_2\times\vec{n}_3)\delta_{\vec{n}_1+\vec{n}_2+\vec{n}_3+\vec{n}_4,0}\,, \tag{10.1}$$

where V is the volume of T^3 and α is a constant. It is known that for any Lie 3-algebra with finite number of generators and positive invariant metric can be reduced to the direct sum of $\mathcal{A}_4$ [23]. Here we have a consistent Lie 3-algebra with positive definite metric while the number of generators is infinite. A natural question is whether our algebra such as (10.1) can be understood as the direct (and infinite) sum of $\mathcal{A}_4$. If this is the case, we need to have a similar quantization condition for the structure constant. It will be very interesting if such quantization of parameter exists in the compact internal space $\mathcal{N}$.

Global structure of internal space. The classification of possible internal manifold $\mathcal{N}$ is another challenging issue. What is required in this paper is that

(i) $\mathcal{N}$ is covered by patches with local coordinates $y^{\dot{\mu}}$ and

(ii) on the intersection of different patches the local coordinates are related with each other by the volume-preserving diffeomorphism.

T^3 is an obvious example of manifold with such structure. In order to understand the relation between M2 and M5, the mathematical classification $\mathcal{N}$ will be indispensable.

Multiple/long M5-brane. In our paper, we construct a single M5-brane action from the BLG model. One of the most challenging issue is how to construct the action of *multiple* M5-branes. For that purpose, we need to construct a set of generators $T^A\chi^a(y)$, where T^A $(A = 1,\dots,d)$ are the generators of an internal algebra and $\chi^a(y)$ is the basis of functions on $\mathcal{N}$. However, as far as we try, it seems difficult to find 3-algebras of this form which satisfies the fundamental identity.

One idea to understand the nature of this problem is to consider the multiple cover of $\mathcal{N}$. Let us take the simplest example T^3 and take all of its radius to 2π for simplicity. Then the basis of functions is of the form $\exp(i\sum_{\dot{\mu}=\dot{1}}^{\dot{3}} n_{\dot{\mu}}y^{\dot{\mu}})$ where $n_{\dot{\mu}}$ is integer. Suppose one takes the double cover in $y^{\dot{1}}$ direction. Then it may be possible to take $n_{\dot{1}}$ to be half integer. So we have two sets of generators, one $\chi^{\vec{n}}$ for $n_{\dot{1}}\in\mathbf{Z}$ and the other $\chi^{\vec{n}}$ for $n_{\dot{1}}\in\mathbf{Z}+1/2$. We write the former generators as $T^{\vec{n}}$ and the latter as $S^{\vec{n}}$. It is then elementary to show that

$$\{T,T,T\}\sim T,\quad \{T,T,S\}\sim S,\quad \{T,S,S\}\sim T,\quad \{S,S,S\}\sim S\,. \tag{10.2}$$

So the 3-algebra of original T^3 is contained in the algebra of covering space as a subalgebra. It is not difficult to show that similar effect occurs in general. Namely let us denote the 3-algebra associated with 3-manifold $\mathcal{N}$ as $\mathcal{A}_{\mathcal{N}}$ and let $\tilde{\mathcal{N}}$ be a covering space of $\mathcal{N}$. Then $\mathcal{A}_{\mathcal{N}}$ becomes a subalgebra of $\mathcal{A}_{\tilde{\mathcal{N}}}$. Since $\mathcal{A}_{\tilde{\mathcal{N}}}$ is not the direct product of $\mathcal{A}_{\tilde{\mathcal{N}}}$ with finite Lie 3-algebra as above, $\mathcal{A}_{\tilde{\mathcal{N}}}$ does not describe multiple M5 but it describes long M5 which wraps $\mathcal{N}$ several times. Such a connection, however, may be helpful to understand the multiple M5 in the future.

Vortex string and volume-preserving diffeomorphism. As we commented, in our construction of M5-brane action, we do not need the metric on $\mathcal{N}$ but only its volume form, or in other words, the 3-form flux C on it. Our computation further implied that it is natural to assume that there is a very large 3-form flux C on the M5 world-volume. This set-up reminds us of the open membrane in large C flux. Since we can neglect the Nambu-Goto part (which contains the metric), the action becomes that of the topological membrane [24],

$$S \sim \int C_{\mu\nu\rho} dX^\mu \wedge dX^\nu \wedge dX^\rho \,. \tag{10.3}$$

When this membrane has the boundary on M5, this topological action gives

$$S \sim \int C_{\mu\nu\rho} X^\mu dX^\nu \wedge dX^\rho \,. \tag{10.4}$$

It gives an action for the string which describes the boundary of the open membrane. When the target space has 3 dimensions and $C \sim \epsilon^{\dot\mu\dot\nu\dot\lambda}$, this action is identical to the kinetic term of the vortex string [25], which was found long ago. In the supermembrane context it was studied in [20−22]. In particular it was found that it can be equipped with the Poisson structure with the constraint associated with the diffeomorphism which defines the volume-preserving diffeomorphism naturally [21],

$$\delta X^{\dot\mu} = \{X^{\dot\mu}, \omega(f,g)\}_D = v^{\dot\mu}(X) + \cdots \,, \tag{10.5}$$

$$v^{\dot\mu} = \epsilon^{\dot\mu\dot\nu\dot\lambda} \partial_{\dot\nu} f \partial_{\dot\lambda} g \,, \quad \partial_{\dot\mu} v^{\dot\mu} = 0 \,, \tag{10.6}$$

$$\omega(f,g) := \int d\sigma f(X) dg(X) \,. \tag{10.7}$$

Here $\{\,,\,\}_D$ is the Dirac bracket associated with the kinetic term and $\cdots$ in the first line describe the extra variation along the world-sheet which can be absorbed by the reparametrization of the world-sheet. In Bagger-Lambert theory, the gauge parameter has an unusual feature that it has two index Λ_{ab}. In this picture, this structure is naturally interpreted as a result of the fact that for the string we can introduce two functions f, g to define the generators on the world-sheet. We hope that this connection with the vortex string would give a new insight into the BLG model.

Acknowledgments

We appreciate partial financial support from Japan-Taiwan Joint Research Program provided by Interchange Association (Japan) by which this collaboration is made possible.

The authors thank Kazuyuki Furuuchi, Darren Sheng-Yu Shih, and Wen-Yu Wen for helpful discussions. P.-M. H. is grateful to Anna Lee for assistance in many ways. The work of P.-M. H. is supported in part by the National Science Council, the National Center for Theoretical Sciences, and the LeCosPA Center at National Taiwan University. Y. M. is partially supported by Grant-in-Aid (#20540253) from the Japan Ministry of Education, Culture, Sports, Science and Technology. Y.I. is partially supported by a Grant-in-Aid for Young Scientists (B) (#19740122) from the Japan Ministry of Education, Culture, Sports, Science and Technology.

A. Derivation of some equations

In this appendix we derive some equations used in the main text. We first derive the commutation relation (4.30). By using the explicit form of the covariant derivative in (4.20), we obtain

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\Phi = -gF^{\dot\kappa}_{\mu\nu}\partial_{\dot\kappa}\Phi, \tag{A.1}$$

where the explicit form of $F^{\dot\kappa}_{\mu\nu}$ in terms of the potential is

$$F^{\dot\kappa}_{\mu\nu} = \epsilon^{\dot\kappa\dot\mu\dot\nu}\partial_\mu\partial_{\dot\mu}b_{\nu\dot\nu} - g\epsilon^{\dot\mu\dot\nu\dot\rho}\partial_{\dot\mu}b_{\mu\dot\nu}\epsilon^{\dot\kappa\dot\lambda\dot\tau}\partial_{\dot\rho}\partial_{\dot\lambda}b_{\nu\dot\tau} - (\mu \leftrightarrow \nu). \tag{A.2}$$

Because the (non-covariant) derivative appears on the right hand side, $F^{\dot\kappa}_{\mu\nu}$ defined by (A.1) is not covariant. We can define the covariantized F by

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\Phi = -g\mathcal{F}^{\dot\kappa}_{\mu\nu}\mathcal{D}_{\dot\kappa}\Phi. \tag{A.3}$$

These two fields are related by $F^{\dot\kappa}_{\mu\nu}\partial_{\dot\kappa}\Phi = \mathcal{F}^{\dot\kappa}_{\mu\nu}\mathcal{D}_{\dot\kappa}\Phi$, and by substituting $\Phi = X^{\dot\mu}$ into this relation and using

$$g\mathcal{D}_{\dot\mu}X^{\dot\sigma} = V\delta^{\dot\sigma}_{\dot\mu}, \tag{A.4}$$

we obtain

$$V\mathcal{F}^{\dot\kappa}_{\mu\nu} = gF^{\dot\lambda}_{\mu\nu}\partial_{\dot\lambda}X^{\dot\kappa}. \tag{A.5}$$

$\mathcal{F}$ can be expressed as the covariant derivative of the field strength $\mathcal{H}$.

$$V\mathcal{F}^{\dot\kappa}_{\mu\nu} = g\mathcal{F}^{\dot\lambda}_{\mu\nu}\mathcal{D}_{\dot\lambda}X^{\dot\kappa} = \epsilon_{\mu\nu\lambda}\epsilon^{\lambda\rho\sigma}\mathcal{D}_\rho\mathcal{D}_\sigma X^{\dot\kappa} = \epsilon_{\mu\nu\lambda}\mathcal{D}_\rho\widetilde{\mathcal{H}}^{\rho\lambda\dot\kappa}. \tag{A.6}$$

In the first step we used the relation (A.4). Substituting this into (A.3), we obtain (4.30).

Next, let us consider the equations of motion of the gauge fields $b_{\dot\mu\dot\nu}$ and $b_{\mu\dot\nu}$. For a variation of $b_{\dot\mu\dot\nu}$, we have the following variations of the action.

$$\begin{aligned}
\delta S_X &= \int d^3x\langle\delta b^{\dot\mu}\mathcal{D}_\mu\mathcal{D}_\mu X^{\dot\mu}\rangle = \frac{1}{2}\int d^3x\langle\delta b^{\dot\mu}\epsilon^{\dot\mu\dot\rho\dot\sigma}\mathcal{D}_\mu\mathcal{H}_{\mu\dot\rho\dot\sigma}\rangle && \text{(A.7)}\\
\delta S_{\rm pot} &= \frac{g^4}{2}\int d^3x\langle\delta b^{\dot\mu}\{X^I, X^J, \{X^I, X^J, X^{\dot\mu}\}\}\rangle, && \text{(A.8)}\\
\delta S_{\rm int} &= \frac{ig^2}{2}\int d^3x\langle\overline{\Psi}, \Gamma_{\dot\mu J}\{\delta b^{\dot\mu}, X^J, \Psi\}\rangle = -\frac{ig^2}{2}\int d^3x\langle\delta b^{\dot\mu}\{\overline{\Psi}\Gamma_{\dot\mu J}, X^J, \Psi\}\rangle, && \text{(A.9)}
\end{aligned}$$

and the equation of motion is

$$\begin{aligned}
0 &= \frac{1}{2}\epsilon^{\dot\mu\dot\rho\dot\sigma}\mathcal{D}_\mu\mathcal{H}_{\mu\dot\rho\dot\sigma} + \frac{g^4}{2}\{X^I, X^J, \{X^I, X^J, X^{\dot\mu}\}\} - \frac{ig^2}{2}\langle\delta b^{\dot\mu}\{\overline{\Psi}\Gamma_{\dot\mu J}, X^J, \Psi\}\rangle\\
&= \frac{1}{2}\epsilon^{\dot\mu\dot\rho\dot\sigma}\mathcal{D}_\mu\mathcal{H}_{\mu\dot\rho\dot\sigma} + \mathcal{D}_{\dot\mu}\mathcal{H}_{\dot1\dot2\dot3} + g^2\epsilon^{\dot\rho\dot\mu\dot\tau}\{X^i, X^{\dot\rho}, \mathcal{D}_{\dot\tau}X^i\}\\
&\quad + \frac{g^4}{2}\{X^i, X^j, \{X^i, X^j, X^{\dot\mu}\}\} - \frac{ig^2}{2}\{\overline{\Psi}\Gamma_{\dot\mu J}, X^J, \Psi\},
\end{aligned} \tag{A.10}$$

or, equivalently,

$$\mathcal{D}_\mu\mathcal{H}^{\mu\dot\rho\dot\sigma} + \mathcal{D}_{\dot\mu}\mathcal{H}^{\dot\mu\dot\rho\dot\sigma} = gJ^{\dot\rho\dot\sigma}, \tag{A.11}$$

where the current is given in the text.

For the variation of the gauge field $b_{\lambda\dot{\mu}}$, we obtain

$$\delta S_X = -g\int d^3x \langle \delta b_{\lambda\dot{\mu}}\{X^I, \mathcal{D}_\lambda X^I, y^{\dot{\mu}}\}\rangle, \tag{A.12}$$

$$\delta S_\Psi = -\frac{ig}{2}\int d^3x \langle \overline{\Psi}\Gamma^\lambda\{\delta b_{\lambda\dot{\mu}}, y^{\dot{\mu}}, \Psi\}\rangle = -\frac{ig}{2}\int d^3x \langle \delta b_{\lambda\dot{\mu}}\{\overline{\Psi}\Gamma^\lambda, \Psi, y^{\dot{\mu}}\}\rangle, \tag{A.13}$$

$$\delta S_{\rm CS} = -\frac{1}{2}\int d^3x \langle \epsilon^{\lambda\mu\nu}\delta b_{\lambda\dot{\mu}} F^{\dot{\mu}}_{\mu\nu}\rangle. \tag{A.14}$$

The equation of motion for $b_{\lambda\dot{\mu}}$ is

$$\frac{1}{2}\epsilon^{\lambda\mu\nu}F^{\dot{\mu}}_{\mu\nu} + g\{X^I, \mathcal{D}_\lambda X^I, y^{\dot{\mu}}\} + \frac{ig}{2}\{\overline{\Psi}\Gamma^\lambda, \Psi, y^{\dot{\mu}}\} = 0. \tag{A.15}$$

This is not covariant, but we can covariantize this by multiplying $g\partial_{\dot{\mu}}X^{\dot{\nu}}$.

$$\frac{V}{2}\epsilon^{\lambda\mu\nu}\mathcal{F}^{\dot{\mu}}_{\mu\nu} + g^2\{X^I, \mathcal{D}_\lambda X^I, X^{\dot{\mu}}\} + \frac{ig^2}{2}\{\overline{\Psi}\Gamma^\lambda, \Psi, X^{\dot{\mu}}\} = 0. \tag{A.16}$$

By using (4.23) and (A.6), we can rewrite this equation of motion as follows:

$$\widetilde{\mathcal{D}}_\rho \mathcal{H}^{\rho\lambda\dot{\mu}} + \mathcal{D}_{\dot{\kappa}}\mathcal{H}^{\dot{\kappa}\lambda\dot{\mu}} = gJ^{\lambda\dot{\mu}} \tag{A.17}$$

The Bianchi identity (5.13) is obtained by substituting $\Phi = X^{\dot{\mu}}$ to the commutation relation (4.29). By using the definition of the field strength $\mathcal{H}$, we can rewrite the left hand side as

$$[\mathcal{D}_\lambda, \mathcal{D}_{\dot{\lambda}}]X^{\dot{\mu}} = \delta^{\dot{\mu}}_{\dot{\lambda}}\mathcal{D}_\lambda\mathcal{H}_{\dot{1}\dot{2}\dot{3}} - \frac{1}{2}\epsilon^{\dot{\mu}\dot{\rho}\dot{\sigma}}\mathcal{D}_{\dot{\lambda}}\mathcal{H}_{\lambda\dot{\rho}\dot{\sigma}}, \tag{A.18}$$

and the right hand side becomes

$$g^2\{\mathcal{H}_{\lambda\dot{\nu}\dot{\lambda}}, X^{\dot{\nu}}, X^{\dot{\mu}}\} = \epsilon^{\dot{\nu}\dot{\mu}\dot{\kappa}}\mathcal{D}_{\dot{\kappa}}\mathcal{H}_{\lambda\dot{\nu}\dot{\lambda}}. \tag{A.19}$$

Combining these, we obtain the Bianchi identity

$$\mathcal{D}_\lambda\mathcal{H}_{\dot{\lambda}\dot{\rho}\dot{\sigma}} - \mathcal{D}_{\dot{\lambda}}\mathcal{H}_{\lambda\dot{\rho}\dot{\sigma}} - \mathcal{D}_{\dot{\rho}}\mathcal{H}_{\lambda\dot{\sigma}\dot{\lambda}} - \mathcal{D}_{\dot{\sigma}}\mathcal{H}_{\lambda\dot{\lambda}\dot{\rho}} = 0. \tag{A.20}$$

This is equivalent to (5.13)

References

[1] P.-M. Ho and Y. Matsuo, *M5 from M2*, *JHEP* **06** (2008) 105 [arXiv:0804.3629].

[2] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108]; *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955]; *Comments on multiple M2-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738].

[3] A. Gustavsson, *Algebraic structures on parallel M2-branes*, arXiv:0709.1260; *Selfdual strings and loop space Nahm equations*, *JHEP* **04** (2008) 083 [arXiv:0802.3456].

[4] V.T. Filippov, *n-Lie algebras*, *Sib. Mat. Zh.* **26** (1985) 126.

[5] Y. Nambu, *Generalized Hamiltonian dynamics*, *Phys. Rev.* **D 7** (1973) 2405;
F. Bayen and M. Flato, *Remarks concerning Nambu's generalized mechanics*, *Phys. Rev.* **D 11** (1975) 3049;
N. Mukunda and E. Sudarshan, *Relations between Nambu and Hamiltonian mechanics*, *Phys. Rev.* **D 13** (1976) 2846;
L. Takhtajan, *On foundation of the generalized Nambu mechanics*, second version, *Commun. Math. Phys.* **160** (1994) 295 [hep-th/9301111];
For a review of the Nambu-Poisson bracket see I. Vaisman, *A survey on Nambu-Poisson brackets*, *Acta Math. Univ. Comenianae* **2** (1999) 213 [math.DG/9901047].

[6] P. Pasti, D.P. Sorokin and M. Tonin, *Covariant action for a $D = 11$ five-brane with the chiral field*, *Phys. Lett.* **B 398** (1997) 41 [hep-th/9701037];
I.A. Bandos et al., *Covariant action for the super-five-brane of M-theory*, *Phys. Rev. Lett.* **78** (1997) 4332 [hep-th/9701149].

[7] M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *World-volume action of the M-theory five-brane*, *Nucl. Phys.* **B 496** (1997) 191 [hep-th/9701166].

[8] P.-M. Ho, Y. Imamura and Y. Matsuo, *M2 to D2 revisited*, *JHEP* **07** (2008) 003 [arXiv:0805.1202].

[9] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[10] Apart from those appearing in other items of the reference list, recent developments on the BLG model include S. Mukhi and C. Papageorgakis, *M2 to D2*, *JHEP* **05** (2008) 085 [arXiv:0803.3218];
M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *$N = 8$ superconformal Chern-Simons theories*, *JHEP* **05** (2008) 025 [arXiv:0803.3242];
D.S. Berman, L.C. Tadrowski and D.C. Thompson, *Aspects of multiple membranes*, *Nucl. Phys.* **B 802** (2008) 106 [arXiv:0803.3611];
M. Van Raamsdonk, *Comments on the Bagger-Lambert theory and multiple M2-branes*, *JHEP* **05** (2008) 105 [arXiv:0803.3803];
A. Morozov, *On the problem of multiple M2 branes*, *JHEP* **05** (2008) 076 [arXiv:0804.0913];
N. Lambert and D. Tong, *Membranes on an orbifold*, arXiv:0804.1114;
J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, *M2-branes on M-folds*, *JHEP* **05** (2008) 038 [arXiv:0804.1256];
U. Gran, B.E.W. Nilsson and C. Petersson, *On relating multiple M2 and D2-branes*, arXiv:0804.1784;
J. Gomis, A.J. Salim and F. Passerini, *Matrix theory of type IIB plane wave from membranes*, arXiv:0804.2186;
E.A. Bergshoeff, M. de Roo and O. Hohm, *Multiple M2-branes and the embedding tensor*, *Class. and Quant. Grav.* **25** (2008) 142001 [arXiv:0804.2201];
K. Hosomichi, K.-M. Lee and S. Lee, *Mass-deformed Bagger-Lambert theory and its BPS objects*, arXiv:0804.2519;
H. Shimada, *β-deformation for matrix model of M-theory*, arXiv:0804.3236;
G. Papadopoulos, *On the structure of k-Lie algebras*, arXiv:0804.3567;
J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert theory for general Lie algebras*, *JHEP* **06** (2008) 075 [arXiv:0805.1012];

S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, *$N = 8$ superconformal gauge theories and $M2$ branes*, arXiv:0805.1087;
A. Morozov, *From simplified BLG action to the first-quantized M-theory*, arXiv:0805.1703;
Y. Honma, S. Iso, Y. Sumitomo and S. Zhang, *Janus field theories from multiple $M2$ branes*, arXiv:0805.1895;
H. Fuji, S. Terashima and M. Yamazaki, *A new $N = 4$ membrane action via orbifold*, arXiv:0805.1997.

[11] M. Schlesinger and J.D. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, *J. Pure Appl. Algebra* **38** (1985) 313;
P. Hanlon and M.L. Wachs, *On Lie k-algebras*, *Adv. Math.* **113** (1995) 206;
J.A. Azcárraga, A.M. Perelomov and J.C. Pérez Bueno, *New generalized Poisson structures*, *J. Phys.* **A 29** (1996) 627 [q-alg/9601007]; *n-ary Lie and associative algebras*, *Rend. Sem. Mat. Univ. Politec. Torino* **53** (1996) 373;
J.A. Azcárraga, J.M. Izquierdo and J.C. Pérez Bueno, *On the higher order generalizations of Poisson structures*, *J. Phys.* **A 30** (1997) L607 [hep-th/9703019];
R. Ibáñez, M. de León, J.C. Marrero and D. Martín de Diego, *Dynamics of generalized Poisson and Nambu-Poisson brackets*, *J. Math. Phys.* **38** (1997) 2332;
P.W. Michor and A.M. Vinogradov, *n-ary Lie and associative algebras*, *Rend. Sem. Mat. Univ. Politec. Torino* **53** (1996) 373 [math.QA/9801087].

[12] R. Weitzenbök, *Invariantentheorie*, P. Noordhoff, Gröningen The Netherlands (1923);
P. Gautheron, *Some remarks concerning Nambu mechanics*, *Lett. Math. Phys.* **37** (1996) 103;
D. Alekseevsky and P. Guha, *On decomposability of Nambu-Poisson tensor*, *Acta Math. Univ. Comenianae* **65** (1996) 1;
R. Ibánez, M. de León, J.C. Marrero and D.M. de Diego, *Dynamics of generalized Poisson and Nambu-Poisson brackets*, *J. Math. Phys.* **38** (1997) 2332;
N. Nakanishi, *On Nambu-Poisson manifolds*, *Rev. Mod. Phys.* **10** (1998) 499;
G. Marmo, G. Vilasi and A.M. Vinogradov, *The local structure of n-Poisson and n-Jacobi manifolds*, *J. Geom. Phys.* **25** (1998) 141 [physics/9709046].

[13] P.S. Howe, G. Sierra and P.K. Townsend, *Supersymmetry in six-dimensions*, *Nucl. Phys.* **B 221** (1983) 331.

[14] L.J. Romans, *Selfduality for interacting fields: covariant field equations for six-dimensional chiral supergravities*, *Nucl. Phys.* **B 276** (1986) 71.

[15] H. Aoki et al., *Noncommutative Yang-Mills in IIB matrix model*, *Nucl. Phys.* **B 565** (2000) 176 [hep-th/9908141].

[16] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[17] K. Kodaira and D.C. Spencer, *On deformations of complex analytic structures, I*, *Ann. Math.* **67** (1958) 328;
K. Kodaira, L. Niremberg and D.C. Spencer, *On the existence of deformations of complex analytic structures*, *Ann. Math.* **68** (1958) 450;
K. Kodaira and D.C. Spencer, *A theorem of completeness for complex analytic fibre spaces*, *Acta Math.* **100** (1958) 281;
K. Kodaira and D.C. Spencer, *On deformations of complex analytic structures, III. Stability theorems for complex structures*, *Ann. Math.* **71** (1960) 43.

JHEP08(2008)014

[18] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219]; *Constrained quantization of open string in background B field and noncommutative D-brane*, *Nucl. Phys.* **B 568** (2000) 447 [hep-th/9906192].

[19] V. Schomerus, *D-branes and deformation quantization*, *JHEP* **06** (1999) 030 [hep-th/9903205].

[20] E. Bergshoeff, D.S. Berman, J.P. van der Schaar and P. Sundell, *A noncommutative M-theory five-brane*, *Nucl. Phys.* **B 590** (2000) 173 [hep-th/0005026];
S. Kawamoto and N. Sasakura, *Open membranes in a constant C-field background and noncommutative boundary strings*, *JHEP* **07** (2000) 014 [hep-th/0005123];
N. Ikeda, *Deformation of BF theories, topological open membrane and a generalization of the star deformation*, *JHEP* **07** (2001) 037 [hep-th/0105286];
B. Pioline, *Comments on the topological open membrane*, *Phys. Rev.* **D 66** (2002) 025010 [hep-th/0201257];
D.S. Berman and B. Pioline, *Open membranes, ribbons and deformed Schild strings*, *Phys. Rev.* **D 70** (2004) 045007 [hep-th/0404049].

[21] Y. Matsuo and Y. Shibusa, *Volume preserving diffeomorphism and noncommutative branes*, *JHEP* **02** (2001) 006 [hep-th/0010040];
M. Sakakibara, *Remarks on a deformation quantization of the canonical Nambu bracket*, *Prog. Theor. Phys.* **104** (2000) 1067.

[22] P.-M. Ho and Y. Matsuo, *A toy model of open membrane field theory in constant* 3*-form flux*, *Gen. Rel. Grav.* **39** (2007) 913 [hep-th/0701130].

[23] The statement first appeared as a conjecture in P.-M. Ho, R.-C. Hou and Y. Matsuo, *Lie* 3*-algebra and multiple M*2*-branes*, *JHEP* **06** (2008) 020 [arXiv:0804.2110];
Later it was proven in G. Papadopoulos, *M*2*-branes,* 3*-Lie algebras and Plucker relations*, *JHEP* **05** (2008) 054 [arXiv:0804.2662];
and J.P. Gauntlett and J.B. Gutowski, *Constraining maximally supersymmetric membrane actions*, arXiv:0804.3078.

[24] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, hep-th/9907026;
J.S. Park, *Topological open p-branes*, hep-th/0012141.

[25] F. Lund and T. Regge, *Unified approach to strings and vortices with soliton solutions*, *Phys. Rev.* **D 14** (1976) 1524.